1

AD-A207 132

ANNUAL REPORT

Chaotic and Bifurcating Nonlinear Systems Driven by Noise with Applications to Laser Dynamics.

For the period: 12/01/87 to 10/31/88

ONR Grant no: N00014-88-K-0084

Submitted to: Office of Naval Research
Code 1112
Arlington, VA 22217

Submitted by: Frank Moss, Principal Investigator
Department of Physics
University of Missouri at St. Louis
St. Louis, MO 63121
Telephone: (314) 553-6150
Bitnet: C4646 @ UMSLVAXA

30/12/88
Frank Moss, Principal Investigator
Professor of Physics

ABSTRACT

The objective of this grant is to pursue research in the field of noise driven nonlinear dynamical systems by means of both analog and digital simulations. Topics for these simulations are normally chosen from contemporary theoretical or experimental works, usually in consultation with the appropriate groups. The emphasis of the research is on laser applications, but other topics, often more generally related to laser dynamics, have also been undertaken. In the area of laser dynamics, two projects have been begun and one is still in progress. The first, is an electronic circuit modelling of the effects of purely multiplicative noise on the correlated spontaneous emission laser, carried out in consultation with W. Schleich (Max-Planck, Munich), H.. Risken and ~~his group~~ (University of Ulm) and M. O. Scully (Max-Planck and University of New Mexico). The second, which is still in progress, is a modelling of the general problem of stochastic resonance, and a study of the effects of the modulation and noise as they relate to the recently observed phenomenon in a dye ring laser. In related areas, four projects have been completed: The first is a generalization of mean-first-passage time calculations to problems involving spatio-temporal noise. Specifically ~~we have~~ considered a Brownian particle moving in a random spatial potential driven by temporal noise, and we highlight an unsolved problem which appears to be extremely difficult, but nevertheless is of wide physical applicability. Second, ~~we have~~ completed an experiment and a contemporary theory on generalized switching processes in the presence of colored noise. Though the object of this work was to study the noise behavior of a generic switch, there is an obvious application to switching in lasers and optically bistable systems. This work was carried out in collaboration with C. Van den Broeck (Limburgs University and the University of Brussels) and P. Hanggi (University of Augsburg). Third, ~~we have made~~ a discovery using analog techniques with strongly colored noise: a noise correlation time induced change in the topology of the two dimensional probability density of multistable systems. The unmistakable implication of this observation for first passage time and bifurcation theory is that the traditional methods which reduce the problem to an "equivalent" one dimensional form are grossly inadequate, and that the multidimensional nature of such problems must be fully accounted for. Fourth, we have completed an interesting example problem on state dependent diffusion posed by Landauer and Van Kampen. While not offering any really new knowledge, it highlights an old question which has frequently been ignored especially in the context of evolution theories. Finally, a considerable effort has been expended in the preparation of a review article on colored noise dynamics which has been solicited by D. ter Haar for Physics Reports. We anticipate that this article will be finished in the Spring or early summer of 1989. The coauthors are: P. Hanggi, P. Jung, and H. Risken. Concurrently, new data acquisition and analysis equipment has been purchased in order to modernize the simulation laboratory; a process which is not yet complete.

CONTENTS

INTRODUCTION

In this report, progress under the grant named on the cover sheet is summarized for the period 12/01/87 to 11/31/88. Six projects were completed in the sense that these were works in six identifiably separate topical areas and that each has lead to a publication. The topical areas spanned a range from first passage time problems to stochastic resonance and state dependent diffusion. They are described in more detail below and in the attached papers in the Appendix.

Each of these projects represented a distinct application of the technique of analog simulation. Specifically, that means that an electronic circuit model of a particular system was constructed, and driven by a noise voltage of well known statistical properties. Measurements of the response of the circuit under various conditions were made and compared to contemporary stochastic theory. Specific, physically observable quantities of interest to the theorists were measured: the first passage time and its probability density; stationary one- and two-dimensional probability densities; time evolving probability densities; and power spectra and correlation functions. Since we are dealing with stochastic systems, each of these quantities must be appropriately averaged.

The experimental process is as follows: An electronic circuit model of the system of interest is designed and built. The "systems" themselves were invariably represented by one or a set of Langevin equations. Noise from a noise generator is passed through a linear filter to establish its correlation time and then applied to the circuit model. The fluctuating responses, i.e. the noisy output voltages, were digitized and then processed by a computer in order to average the quantities of interest. These measurements are then analyzed, the error bars established, and then compared to theory in the same way that experimental data is treated. In this way, analog measurements frequently point the way for experiments on the actual physical systems and can often identify any pitfalls which might arise in the course of the measurements. This latter characteristic has been especially useful in the experiments on stochastic resonance where instrumental band width played a crucial role and was not immediately recognized in either the experiments or in previous digital simulations.[1]

The remainder of this report is organized as follows: the description of the six completed projects is followed by a list of the personnel who have worked on the projects and the degrees granted based on research work completed under the grant. Certain University contributions to this research as well as other grant support is outlined. Then follows an outline of projects currently in progress. The special problems in which we have an interest and the objectives of these current projects are discussed. After this we outline our plans for the second year of the grant. Two areas here seem to be particularly attractive and timely. As is our usual practice, discussions with the appropriate theorists have taken place or are planned. It is important to emphasize, though, that I work in a rapidly developing field where certain "targets of opportunity" sometimes rapidly appear. The projects on stochastic resonance and noise correlation time induced topologies were just such examples. It is, therefore, not prudent to propose a rigid structure for our research. The last section contains an outline of our budget expenditures for both personnel and equipment and our projections of these for next year. Finally, reprints and preprints of papers which were produced with support of this grant are attached in the Appendix.

SUMMARY OF RESEARCH PROJECTS COMPLETED

1) **First passage time with spatio-temporal noise.** The mean first-passage time has for many years been a very useful concept for studies on a wide variety of problems from chemical reaction rates to theories of evolutionary biology. Often interesting applications are found in various types of disordered systems, yet in only one previous instance has an attempt been made to evaluate the MFPT in a realistic random spatially distributed potential.[2] By realistic, we mean a non-discontinuous random functional, as would be found in macroscopic natural systems, in contrast to the usual distribution of delta functions[3,4]. Here we consider a classical Brownian particle moving in a one-dimensional random potential U(x) which has a non zero correlation length ℓ, that is, colored spatial noise. The dynamics are given by the Langevin equation,

$$\dot{x} = -dU(x)/dx + \xi(t), \tag{1}$$

where the correlations of the random functions are given by

$$\langle U(x)U(x')\rangle = (C/\ell)\exp-|x - x'|/\ell, \tag{2}$$

and

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t - t'). \tag{3}$$

Note that the spatial noise of intensity C given by Eq. (2) has a correlation length ℓ, while the temporal noise of intensity D given by Eq. (3) is white.

In spite of intense effort by the theorists: A. Engel, L. Schimansky-Geier and P. Hanggi, all of whom worked on this problem to some extent, the more realistic problem - colored spatial *and* colored temporal noise - could not be effectively treated in any known approximation. We highlight this as an open problem in the published version[5]. The white temporal noise problem which we did pursue was resolved into four expansions, framed in terms which are easy to interpret and apply by experimentalists, which apply in various regimes of parameter space: small or large temporal noise intensity compared to the spatial noise intensity in the two length regimes: small or large separation between the exit boundaries compared to the spatial correlation length. The Einstenian relation for Brownian motion, $\langle x^2\rangle \rightarrow Dt$, is recovered in all appropriate limits. A discussion of the possibilities for observing the predicted higher order departures from the Einstein relation is included. The details can be found in the reprint in the Appendix.

Though considerable effort was expended on analog simulations of this problem, no reliable results were obtained, so only the theory paper was published. The reason is that we were unable to obtain in a reasonable amount of experimental time a suitable ensemble average over sufficiently many realizations of the spatial potential. We have ideas for advancing this technology, i.e. to significantly speed the spatial average over realizations, which, however require the new data analysis systems just now being installed. It should be noted, however, that this work resulted in an extremely practical advance in analog simulation technology: the ability to simulate any **arbitrary spatial potential** in both one and two dimensions. This means that we are no longer limited to simulations of Langevin equations with polynomials made up of powers or trigonometric functions of x and y, but can simulate truly arbitrary functions so long as they are finite.

2) **Noise correlation time induced topologies.** This project was completed in collaboration with F. Marchesoni of the University of Perugia, and it represents an actual discovery which subsequently prompted a degree of theoretical activity. The observations were made initially on a special "soft" bistable potential which has been used in soliton-soliton interactions:

$$U(x) = -\ln\{\mathrm{ch}x/(\mathrm{ch}^2x + \mathrm{sh}^2x)\}, \tag{4}$$

which increases only linearly for sufficiently large x. [It should be noted that this potential is illustrative of our present ability to simulate an arbitrary function as outlined in the previous section.] We simulated the system,

$$\dot{x} = -dU/dx + \xi(t), \tag{5}$$

with strongly colored noise,

$$\langle\xi(t)\xi(t')\rangle = (D/\tau)\exp-|x - x'|/\tau. \tag{6}$$

"Strongly colored", means that the dimensionless correlation time τ, which measures how fast the noise fluctuates on the time scale of the electronic model, is >> 1 (it was typically 3 to 7 depending on the depth of the wells and the strength of the noise D). We measured the joint probability density $P(x,\xi)$. Previous studies using the standard quartic bistable potential, $U = -(1/2)x^2 + (1/4)x^4$, had shown characteristic colored noise topological features[6], notable the so-called "skewing effect" which were simultaneously confirmed by Risken's group using matrix continued fraction techniques[7]. Though the shape of the cross sections through these densities continuously evolved with increasing τ, the topology remained the same. Specifically, the two maxima (corresponding to the two potential wells) were always separated by a single saddle point which was always located on the x axis (at the origin for the standard quartic). The measurements on the soft potential, however, showed a striking new feature: at a certain critical value of τ, the single saddle point "exploded" into two symmetrically oriented but skewed saddles located well off the axis[8].

Confirmation of this phenomenon was soon obtained for an entirely different potential (in fact a periodic potential) by the Risken group. And later the same phenomenon was observed in analog simulations of the standard quartic potential and even of a random potential[9]. So why had this feature not been previously observed in either digital or analog simulations or with the matrix continued fractions? The reason is that a shallow well depth (as was the case for the "soft" potential) is required to observe the effect for moderate values of both D and τ.

The implications of this discovery for mean first passage time theory are significant. Extant theoretical approximations for colored noise, the so-called "small τ" theories, had always reduced the necessarily higher dimensional Fokker-Planck equation to an "equivalent" one-dimensional form.[10] This amounts to choosing a most probable path for the motion of a particle from one well to the other which is a straight line crossing the saddle point at the origin. The measured topologies for τ greater than the critical value demonstrated, in contrast, that the most probable path which now must pass through one of the off-axis saddles was a wide curve and sometimes a tortuous path which, however, always avoided the origin. Obviously, in order to treat first passage time problems for strongly colored noise a multidimensional approach is necessary. This has recently been developed by Matkowsky[11]. In any case, the observations present an interesting problem in bifurcation theory which has stimulated the

interest of several groups[12].

3) **Generic switching in the presence of noise.** Noise necessarily affects all macroscopic (and probably most microscopic) switches. As computer memory element densities tend toward ever larger numbers, with a concomitant reduction in the size of the individual switch elements toward dimensions limited by quantum noise, the subject of noise immunity becomes increasingly important. Moreover, switches operating even at the quantum level are better described with colored noise theories[13].

In this project and the following one, we have undertaken to study: first, the effects of colored noise on a classical generic switch; and second, the predicted effect of noise color on a quantum system, the correlated spontaneous-emission laser (CEL). In both cases, it has only recently become possible to handle the problem of colored noise driven switching with adequate approximate theories[14,15].

In this, the classical, project, we have considered a generic switch as represented by a general Langevin equation of the form,

$$\dot{x} = x[-1 + A(t)/(1 + x^2)] + \xi(t) \tag{7}$$

driven by the usual colored noise

$$\langle \xi(t)\xi(t')\rangle = (D/\tau)\exp-|t - t'|/\tau. \tag{8}$$

The forcing as shown in Eq. (7) may seem strange, but it is the result of the simplest "soft" potential which is still nonlinear and bistable could be written:

$$U(x) = (1/2)[x^2 - A(t)\ln(1 + x^2). \tag{9}$$

[A soft potential is one for which the restoring force increases only linearly as $x \rightarrow \infty$.] Switches with "harder" potentials can easily be represented, but the generic switch should be represented by a soft potential for which the effects of the noise will be greatest. Moreover, this potential is a one-dimensional representation for the dynamics of a ring laser. In the above expressions, $A(t)$ represents the switch signal:

$$A(t) = A_0[-1 + 2u(t)], \tag{10}$$

where $u(t)$ is the Heavyside (or unit step) function. Whence, $A(t) = -A_0$ for all $t < 0$ and $A(t) = +A_0$ for all $t >= 0$. What this really means is that the potential is switched (in a time short compared to all other time scales in the simulation) at $t = 0$ from a potential which is globally stable at $x = 0$ to one which is bistable. The colored noise $\xi(t)$ is always present, and forms a random initial condition which drives the switch off the unstable state (on top of the barrier separating the stable states for $t > 0$).

The probability density and the first moment of the time necessary for the switch to respond by crossing a predetermined threshold were measured and calculated. This is a generalization of the more frequently encountered (but less physically realistic) problem in the decay of an unstable state wherein the initial condition is taken as a delta function and some mean first passage time theory is applied. The general approximate theoretical framework for this problem has been developed by Dhara and Menon[14], and we have applied it specifically to the generic switch[15]. For more details, see the paper reproduced in the Appendix[15].

More recently, Sancho and San Miguel and their coworkers, stimulated by this work, have reworked and expanded on the same problem with an improved theory[16,17]. We regard this body of work as a significant addition to the literature on the subject of the decay of unstable and meta stable states or more specifically: noisy switching processes.

4) **Noise quenching in simulations of the correlated spontaneous-emission laser (CEL).** This project is a natural outgrowth of the previous one. Though we do not consider switching in this simulation, we do test some predictions regarding noise quenching in the CEL both via analog modelling and by matrix continued fraction techniques. Colored noise theory is only just beginning to be applied to quantum systems[18]. The colored noise theory seems to be difficult, since for this particular model the noise is both colored and multiplicative. The dynamics of the quantum system can, however, be reduced to a description by a classical Langevin equation[13]:

$$\dot{\psi} = a - b\sin\psi + \xi(t)\sin(\psi/2) \tag{11}$$

with the usual colored noise given by Eq. (8). In this case, it was a discovery of the analog simulation that noise color (that is to say, increasing noise correlation time) **further enhances the noise quenching** already resulting from squeezing in the CEL. This result has since been corroborated by matrix continued fraction methods and by a quantum colored noise theory[19], and may have significant practical implications for the design of reduced quantum noise measuring devices[20]. For further details, see preprint in the Appendix.

5) **Stochastic resonance.** This project is an example of how rapidly analog simulation can respond to arising topics of current interest. Stochastic resonance (SR) is the name given to the phenomenon of noise assisted transitions in bi- or multi-stable systems. It is a purely nonlinear effect whereby the signal-to-noise ratio of a switching signal can be **considerably enhanced by the deliberate addition of noise.** It fits very well indeed into our program along with the two previous projects on the study of noisy switching systems. SR was advanced about a decade ago as a possible explanation of the observed periodicity in the recurrences of the Earth's ice ages[21,22]. An interesting experiment wherein SR was observed in a dye ring laser has recently refocused attention on the problem[23]. I only learned of the laser experiment from R. Roy at the Los Alamos workshop[9] in March 1988, well before his Phys. Rev. Letter[23] was written. It was nevertheless possible to complete an analog simulation of this system by early summer and to prepare a publication[1]. While the laser system is multidimensional (at least two dimensional), the basic phenomenology of SR can be represented by just two Langevin equations: one which represents additive modulation,

$$\dot{x} = x - x^3 + \epsilon\cos\omega t + \xi(t) \tag{12}$$

and the other representing multiplicative modulation,

$$\dot{x} = [1 + \epsilon\cos\omega t]^{1/2}x - x^3 + \xi(t), \tag{13}$$

where the $\cos\omega t$ term represents the periodic modulation and $\xi(t)$ is the noise. The forcing, $x - x^3$, is just that due to the "standard" quartic potential, $U(x) = -(1/2)x^2 + (1/4)x^4$. In the first case, both the modulation and the noise are additive. This means that the potential is "tipped" up and down by the modulation so that the potential wells alternately are raised and lowered in relation to the barrier separating them. In

the second case, the wells remain stationary while the height of the barrier is modulated.

In both cases, the probability that a transition from one well to the other will occur is also a periodic function of time. Thus the problem is non stationary, and hence raises interesting theoretical challenges[24-28]. What had escaped previous notice in both digital simulations and in experiments was the fact that the power spectrum of this system (at least that of Eq. (12)) must contain a delta function at the modulation frequency. That observation is crucial for all experiments and simulations. It means that efforts to measure the signal-to-noise ratio by measuring the amplitude of the peak in the power spectrum can at best be only qualitative. This observation prompted our comment[1]. In our simulation, the "input" and "output" signal-to-noise ratios were measured for the two systems represented by Eqs. (12) and (13). Our results were compared with the McNamara and Wiesenfeld theory[25]. For more details see preprint in the Appendix.

6) **State dependent diffusion**. This simulation did not contribute any new knowledge but rather was undertaken at the request of Rolf Landauer in order to illustrate and emphasize an old point: that the location of the point of global stability in a multi stable system can be changed with multiplicative (or "state dependent") noise. We chose an interesting example for which three independent theoretical approaches had been taken.[29-31] For this example, a standard quartic potential was chosen, with the noise intensity everywhere constant except at one discontinuous "hot spot" near the potential maximum but located on one side of the barrier. It was something of a challenge to simulate this "state dependent" noise with electronic circuits. The instantaneous location of the trajectory was sensed with comparators and sample-and-hold circuits, while some "extra" noise was switched in when the trajectory was located within the "hot spot". We mad measurements of the stationary probability densities as a function of the magnitude of the extra noise in the hot spot. Our results were compared to the theories of Landauer and of Van Kampen (except for a differing definition of particle velocity, the results are equivalent, though the approaches are quite different: Van Kampen's being totally thermodynamical while Landauer's was totally statistical.) Together these three papers[29,31,32] make an interesting pedagogical collection on the particular point regarding global stability.

Buttiker's paper[30], on the other hand, has more practical consequences. He predicts a "noise induced probability current" in a continuous, spatially periodic one-dimensional system with periodically placed hot spots. Thus a mechanical current can be the result of spatially periodic heating. There may be applications of this theory to the technology of superlattice fabrication. We have also simulated this system, and observed and measured Buttiker's probability currents, but have not yet written up those results for publication.[33]

PERSONNEL, CONSULTANTS AND DEGREES GRANTED

1) **Students.** Two graduate students were supported during one summer and partially supported during one academic year. They both received MS degrees in August 1988. Their names and thesis topics are:

Mr. Goutam Debnath, "Observation of stochastic resonance in a bistable potential as a model for a bidirectional ring laser." MS thesis; University of Missouri at St. Louis; July 1988.

Miss Kathakali Sinha, "State dependent diffusion (SDD) in a bistable potential". MS thesis; University of Missouri at St. Louis; July 1988.

Unfortunately, these two students did not elect to remain at this University to pursue PhD work.

In addition, an undergraduate student Mr. Gabor Schmera was supported part time during the summer and academic year.

2) **Visitor.** Professor Ting Zhou of the Institute for Semiconductors, The Chinese Academy of Sciences, Beijing has been a visitor from May 1988 to May 1989. He has been supported completely by funds made available from the University of Missouri. He has worked full time on the projects described herein.

3) **Consultants.** My grant provided for theoretical consultants. The following consultants, listed along with the topics they were consulted on, were supported to some extent by the grant:

a) Prof. H. Risken, University of Ulm, FRG
Noise correlation time induced topologies.

b) Prof. P. Mandel, Free University of Brussels, Belgium
Noise driven bifurcations in lasers. Chaotic responses of coupled CO_2 lasers.

c) Prof. P. Hanggi, University of Augsburg, FRG
Stochastic resonance. Review of colored noise in double wells: an invited review article in preparation for Physics Reports.

d) Dr. W. Schleich, Max-Planck Institute for Quantum Optics, Munich
Noise quenching in correlated spontaneous-emission lasers.

e) Dr. C. Van den Broeck, Limbergs Universitair Centrum and University of Texas at Austin.
Random potentials. Decay of an unstable state.

f) Prof. J. M. Sancho, University of Barcelona, Spain
Decay of an unstable state driven by colored noise.

UNIVERSITY CONTRIBUTIONS TO THIS RESEARCH PROJECT

Before this grant was awarded the University of Missouri committed matching support to a maximum of $80,000. To date only $13,000 of this has been made available. That money has gone entirely for the support of Professor Zhou as mentioned above. I have been assured, however that the full amount promised will be made available.

PROJECTS CURRENTLY IN PROGRESS

1) **Noise correlation time induced topologies.** This project has resulted in one Letter publication[8] and one poster[9], but is far from complete. In collaboration with the Risken group at Ulm, a complete study of this new critical feature is being carried out over all parameter ranges. We expect this work to be finished in the spring or early summer of 1989.

2) **Stochastic resonance.** This project has resulted in one Comment which is also letter length[1], but is also not complete. In addition to signal-to-noise ratios, that is power spectra, we hope to measure a different quantity which must also be sensitive to the correlation between the output and the modulation, the density of residence times. This quantity has never been measured in an SR system nor is there yet a theory for it. It is a quantity which is sensitive to time correlated changes in the transition probability. In addition, this quantity does not suffer from the defect of the power spectra in that there should be no delta functions in the output. We also wish to measure the total power in the band of both the signal and the noise in order to test a remarkable prediction of the McNamara-Wiesenfeld theory[25]: that the the signal actually absorbs power from the noise near the maximum in the signal-to-noise ratio curve. It has to date been impossible to test this prediction in the laser experiment.

3) **Review article for Physics Reports.** As stated above, this is a review to be coauthored by Moss, Jung, Hanggi and Risken. We expect it to be finished by early or mid summer 1989.

4) **Time evolution of probability densities in oscillators with weak nonlinearities.** We are just at the beginning of this project. The appropriate Langevin equation is

$$\ddot{x} + \gamma\dot{x} + kx - \epsilon x^3 = 0, \tag{14}$$

where γ is the damping, k the spring constant and ϵ is a (weak) nonlinearity parameter. In the case that the damping is noisy, $\gamma \rightarrow \gamma_0 + \xi(t)$, the linear ($k = 0$) version of this oscillator is known to explode (all moments $\rightarrow \infty$) for some value of the noise intensity. It is not known how weak nonlinearity can affect this behavior. Moreover, for the first time ever, the time evolution of the probability density P(x,t) has been obtained exactly for the linear case. This result is due to J. Wm. Turner and R. Lefever of the University of Brussels and has not yet been written up for publication. We have already constructed a simulator of this oscillator and begun the measurements of P(x,t). The oscillator with noisy damping is itself not physically interesting, but the explosion can be viewed as a universal bifurcation event. (The usual normal form analysis in conventional bifurcation theory always begins with a linearization, valid for small parameters, near the bifurcation point.) The recent solution of the time dependent Fokker-Planck equation for this linear system, may lead to a truly nonlinear bifurcation theory.

5) **Postponements and advancements of noisy Hopf bifurcations.** This is a topic which we have been interested in for some time. We have previously done experiments on both the stationary[34] and time dependent[35] properties of the Hopf bifurcation in the Brusselator. Of interest is whether noise can produce an advancement or a postponement of the Hopf bifurcation. The original analog simulation in the Brusselator showed only a postponement[34]. One theory predicts only postpone-

ments[36], while two others predict advancements[37,38]. In spite of our earlier measurements, we are gathering evidence, from precision digital simulations, that there are indeed advancements. It is possible that the advancements, if present in the earlier analog simulations were not visible due to a lack of precision combined with the fact that the Brusselator limit cycle is highly irregular. The digital simulations and the theory are currently being applied to the Hopf bifurcation in the Poincare' oscillator:

$$\ddot{x} + [A + B(x^2 + \dot{x}^2)]\dot{x} + \gamma x + \xi(t) = 0 \qquad (15)$$

where the noise can either be additive, as shown is the above equation, or multiplicative, in which case $B \rightarrow B_0 + \xi(t)$. This oscillator has the advantage that the limit cycle is regular, i.e. it is a circle, and the magnitude of the velocity of the trajectory $\dot{x}(x) \cong$ const. This question is important in the analysis of real, macroscopic, physical systems which are always noisy. Are bifurcations advanced or postponed by this noise? And if so, how does the magnitude of the advancement (postponement) depend on the ratio of the noise correlation time and the limit cycle period? We do not have good answers to these questions yet either from the experimental or theoretical points of view or from simulations. Digital data are slow in coming. Near the bifurcation point very large amounts of CPU time (the order of hours) on our VAX 8600 is required just to obtain one two-dimensional probability density $P(x,\dot{x})$. The analog simulations are much faster (order of minutes) but of less precision.

PLANS FOR THE SECOND YEAR

Obviously, the first priority is to finish the current projects listed above. However, there are two additional projects which we would like to pursue time and supply of students permitting.

1) **Weak chaos: weakly dissipative, weakly nonlinear oscillators.** This project has not taken form yet but the idea is to examine nearly Hamiltonian systems using our electronic modelling techniques. We are following here the work of G. M. Zaslavsky and his coworkers regarding this possibility. He has observed the initial stages of the breakup of KAM tori at the onset of Hamiltonian chaos in various weakly driven oscillators. Using digital simulations, we have been able to reproduce some of his results, but the requirements on CPU time are enormous. Therefore, if it is possible to study any of the phenomenology using analog simulations this would be of great interest. In particular, Zaslavsky has observed the formation of "stochastic webbs" for very weak chaos. These are just chaotic phase plane trajectories close to the separatrices, for example in extended systems with periodic potentials which are driven periodically[39-41]. Sometimes the webbs display interesting symmetries, e. g. quasi crystal symmetries. An example is the following:

$$\ddot{x} + \omega_0^2 \sin x = \epsilon \sin(kx - \omega t) \qquad (16)$$

For rational ratios ω/ω_0, and for certain (small) values of ϵ, the webb is formed around the phase plane trajectories $\dot{x}(x)$. Now we have made simulations of similar, though non-Hamiltonian, systems:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 \sin x = \xi(t) \qquad (17)$$

and we have been able to approach Hamiltonian behavior for $\gamma \rightarrow 0$. The noise $\xi(t)$, then drives the system, and for very small γ only a very small noise intensity is nec-

essary to keep the system on a trajectory near the Hamiltonian separatrix. In this case, we have measured the two-dimensional probability density $P(x,\dot{x})$. The contours of constant probability in the phase plane then look very much like the separatrix. We call these densities the stochastic phase portraits[42]. In fact they look very much like Zaslavsky's stochastic webbs, but of course our system is not chaotic being driven by (weak) high dimensional noise instead of a periodic time function. It certainly is easy to build an analog simulator of Eq (16), albeit with the added damping term $\gamma\dot{x}$. The question is, by careful design is it possible to make γ small enough to unmistakably observe the chaotic webbs? Alternatively one can ask if any of the symmetries observed by Zaslavsky are preserved when high dimensional noise drives the system? The actual system to be simulated would be represented by:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 \sin x = \epsilon \sin(kx - \omega t) + \xi(t). \tag{18}$$

It is probable that what would be observed would be some mixture of the chaotic webbs and the noise driven portraits with $\xi(t)$ being the smallest possible inherent electronic noise. Could these yield any useful information? The zaslavsky group have agreed to collaborate with me on these simulations.

[For the record Michael, we contacted the Zaslavsky group nearly 8 months ago, based on one of their publications in Physics Letters, long before the Physics Today article and before I had seen the Nature article. They are also writing a book for our Cambridge Nonlinear Dynamics series.]

2) **Coupled Oscillators: a model for two coupled lasers.** This is a three way loose collaboration between myself and P. Mandel (Brussels) who is doing the theory and P. Glorieux (Lille) who is doing the experiment with coupled CO_2 lasers. They would like to have a "catalogue of the phenomenology" from us in order to point them toward regions in parameter space where interesting behavior is expected. The system is five-dimensional, and is expected to exhibit complex behavior. We have previously modeled a single laser which was driven by an external oscillator (instead of a second laser) and observed a rich dynamics, including a Farey Tree of phase locked regions, slow mode modulational instabilities (later observed in real lasers), Arnold tongues and transitions to chaos[43], so the extension to two coupled lasers would probably be straight forward. The single laser model is:

$$\dot{E} = -kE + (g^2/\gamma_0)EN \tag{19a}$$

$$\dot{N} = -\gamma_1(N - N_0) - (4g^2/\gamma_0)|E|^2N \tag{19b}$$

with periodically modulated losses

$$k \rightarrow k_0[1 + m\cos(\omega t)], \tag{20}$$

where E is the field intensity, N is the population inversion, the γ are the two polarization modes, g is the atomic field coupling factor, and k is the cavity loss. The proposed coupling of two lasers would be accomplished by coupling the cavity losses, so that $k_1 = bk_2$, where the subscripts now represent laser 1 and laser 2. The system is five-dimensional with two variables for each in E, N, and k.

I wish to add a caveat here: this project is expected to be long and difficult. It will necessitate a very careful cataloging of the expected wide variety of complex dynamic behavior. I cannot complete this and also meet my other commitments with-

out either some new (and very good) students or a postdoctoral student, preferably both. In view of the University's commitment outlined above, money is not a problem, however we have had difficulty in attracting good graduate students here recently.

3)**Targets of opportunity**. It is necessary to save both some time and laboratory capacity in order to respond to interesting situations which may arise. During this past year the project on stochastic resonance was just such an opportunity.

REFERENCES

1. G. Debnath, T. Zhou and Frank Moss, Phys. Rev. A. submitted. See preprint in Appendix.

2. P. de Gennes, J. Stat. Phys. **12**, 463 (1975)

3. see for example, S. Alexander, J. Bernasconi, W. R. Schneider and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981)

4. S. Havlin, J. Kiefer and G. H. Weiss, Phys. Rev. A **36**, 1403 (1987)

5. A. Engel and F. Moss, Phys. Rev. A Rapid Comm. **38**, 571 (1988). See preprint in Appendix.

6. F. Moss and P. V. E. McClintock, Z. Phys. B. **61**, 381 (1985)

7. P. Jung and H. Risken, Z. Phys. B. **61**, 367 (1985).

8. F. Marchesoni and F. Moss, Phys. Lett. **A131**, 322 (1988) see preprint in Appendix.

9. These later results are as yet unpublished, but a preliminary report was given at the Workshop on Noise and its External Interaction with Spatial Degrees of Freedom in Nonlinear Dissipative Systems, Los Alamos, NM; March 1988; proceedings to be published in J. Stat. Phys.

10. See for example the review, K. Lindenberg and B. West in *Noise in Nonlinear Dynamical Systems*, Vol. 1, edited by Frank Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, forthcoming January 1989)

11. M. M. Dygas, B. J. Matkowsky and Z. Schuss, SIAM J. Appl. Math. **48** April 1988.

12. The exact bifurcation theory is evidently challenging. An approximate theory has been completed: P. Jung and P. Hanggi, Proc. Workshop on Noise and its External Interaction..., Los Alamos, NM, March 1988; J. Stat. Phys., to be published. Other groups which have attempted calculations are L. Schimansky-Geier (Berlin) and J. M. Sancho (Barcelona), private communications.

13. See for example the discussions of colored quantum noise in: M. James, F. Moss, H. Risken and W. Schleich, Phys. Rev. A. to be published; see preprint in Appendix. For the application to the CEL Laser see: J. Bergou, M. Orzag and M. O. Scully Phys. Rev. A **38**, 768 (1988) and references therein. For the white noise CEL Laser theory, see: W. Schleich and M. O. Scully, Phys. Rev. A. **37**, 3010 (1988) and ibid, 1261 (1988), and references therein.

14. A. K. Dhara and S. V. G. Menon, J. Stat. Phys., **46**, 743 (1987)

15. M. James, F. Moss, P. Hanggi and C. Van den Broeck, Phys. Rev. A. **38**, 4690 (1988) see preprint in Appendix.

16. J. M. Sancho and M. San Miguel, Phys. Rev. A, to appear.

17. P. Colet, M. San Miguel, J. Casademunt and J. M. Sancho, Phys. Rev. A., to appear.

18. M. O. Scully, private communication and invited lecture: Conference on Noise and Chaos in Nonlinear Dynamical Systems; Turin, Italy, March 1989; Proceedings to be published by Cambridge University Press; F. Moss, L. A. Lugiato, W. Schleich and P. Grigolini, editors.

19. H. Risken and W. Schleich, private communication and to be published; see also,M. James, F. Moss, W. Schleich and H. Risken, Phys. Rev. A., to be published, preprint in Appendix.

20. Quantum noise reduction is the underlying theme of the Conference on Noise and Chaos in Nonlinear Dynamical Systems; Turin, Italy, March 1989.

21. C. Nicolis and G. Nicolis, Tellus **33**, 225 (1981)

22. R. Benzi, A. Sutera and A. Vulpiani, J. Phys. A. **14**, L453 (1981)

23. B. McNamara, K. Wiesenfeld and R. Roy, Phys. Rev. Lett., **60**, 2626 (1988)

24. The first successful theory of SR was an adiabatic (or quasi stationary) approximation to the appropriate Fokker-Planck Equation: C. Nicolis, Tellus **34**, 1 (1982).

25. B. McNamara and K. Wiesenfeld, Phys. Rev. A. preprint. This theory is similar in spirit to the Nicolis theory in that it is an adiabatic approximation. The elaboration is, however, from the point of view of the signal-to-noise ratio of a general nonlinear system.

26. P. Jung and P. Hanggi, EuroPhys. Lett., preprint. These authors have an exact solution which was made possible by an interesting "trick", i.e. averaging over a random phase ϕ, in the modulation term: $\epsilon\cos(\omega t + \phi)$.

27. R. Fox, Phys. Rev A (?) preprint. This author has made a colored noise theory for SR.

28. F. Marchesoni, et. al. Phys. Rev. Lett. (?) preprint. These authors also have done an analog simulation of the SR problem.

29. R. Landauer, **Proc. Workshop on Noise and its Interaction with External Degrees of Freedom in Nonlinear Dissipative Systems**, edited by C. Doering, R. Deissler and H. Brand; J. Stat. Phys., to be published.

30. M. Buttiker, *loc. cit.* Ref. 29.

31. N. G. Van Kampen, IBM J. Res. & Dev. 32, xxx (1988).

32. K. Sinha and Frank Moss, *loc. cit.* Ref. 29; see preprint in Appendix.

33. T. Zhou and Frank Moss, unpublished.

34. L. Fronzoni, F. Moss, R. Mannella and P. V. E. McClintock, Phys. Rev. A **36**, 834 (1987)

35. L. Fronzoni, F. Moss and P. V. E. McClintock, Phys. Rev. A. **36**, 1492 (1987).

36. V. Altares and G. Nicolis, Phys. Rev. A **37**, 3630 (1988)

37. R. Lefever and J. Wm. Turner, Phys. Rev. Lett. **56**, 1631 (1986)

38. K. Lekkas, L. Schimansky-Geier and H. Engel-Herbert, Z. Phys. B **70**, 517 (1988).

39. See for example, A. A. Chernikov, R. Z. Sagdeev, D. A. Usikov, M. Yu. Zakharov, and G. M. Zaslavsky, Nature **326**, 559 (1987)

40. G. M. Zaslavsky, **Chaos in Dynamic Systems**, Harwood Academic, New York (1985)

41. A. A. Chernikov, R. Z. Sagdeev and G. M. Zaslavsky, **Physics Today**, November (1988) p. 27.

42. F. Moss, P. Hanggi and P. V. E. McClintock, Phys. Rev. A. **33**, 4459 (1986)

43. Mark James and Frank Moss, JOSA-B, special issue on the **Nonlinear Dynamics of Lasers**, edited by, D. Bandy, A. N. Oraevsky and J. R. Tredicce, 5, 1121 (1988).

BUDGET EXPENDITURES AND PROJECTIONS

AM090 SUMMARY STATEMENT IN WHOLE DOLLARS AS OF 10/31/88 PAGE 1

ACCOUNT NUMBER S-5-32725 N00014-88-K-0084 MOSS 10/88

F MOSS
527 BENTON HALL

P.I. - F MOSS

BUDGET PERIOD:
START - 12/01/87
END - 10/31/88
OH RATE/BASE - 46.00% MTDC

SUB CODE	DESCRIPTION	ORIGINAL BUDGET	CURRENT BUDGET	CURRENT MONTH	PROJECT TO -DATE	ENCUMBRANCES	REQUISITIONS	BALANCE AVAILABLE
2000	BUDG POOL SAL&WAGES	28,627	8,054					8,054.00
2100	RANKED FACULTY		14,733		9,822.00			4,911.00
2170	ACADEMIC ASSISTANTS		5,840	300	5,140.00	700		0.00
	*PERSONAL SERVICES	28,627	28,627	300	14,962.00	700		12,965.00
2800	STAFF BENEFITS	5,797	5,797	1	1,676.70			4,120.30
3000	EXP & EQUIP BUDG POO	54,581	39,444					39,443.62
3100	TRAVEL-DOMESTIC	4,000	2,805		1,629.30			1,175.87
3150			1,195		1,194.83			0.00
4700	PUBLISHING		1,392		1,017.00	375		0.00
4800	REPRODUCTION COST		135	45	134.91			0.00
5100	SUPPLIES/SERVICES		6,680	618	6,377.17	303		0.00
5500	CONSULTING SERVICES		4,860	1,980	4,860.00			0.00
5550			2,070		2,070.00			0.00
7700	EQUIPMENT	112,212	112,212	2,263	8,592.60	9,674		93,945.90
	*OTHER DIRECT COSTS	176,590	176,590	4,907	27,552.51	10,352		138,685.69
	**DIRECT COSTS-GRANTOR	205,217	205,217	5,207	42,514.51	11,052		151,650.69
9800	GRANTOR IDC	42,782	42,782	1,354	15,604.08			27,177.92
	****TOTAL COSTS-GRANTOR	247,999	247,999	6,561	58,118.59	11,052		178,828.61

OPEN COMMITMENTS STATUS

ACCOUNT	REF.	DATE	DESCRIPTION	ORIGINAL AMOUNT	LIQUIDATING EXPENDITURES	ADJUST-MENTS	CURRENT AMOUNT	
S-5-32725-2170	999999	09/30	SALARY ENCUMBRANCE	1,000.00	300.00		700.00	
S-5-32725-4700	S73842	06/20	AMERICAN PHYSICAL	375.00			375.00	
S-5-32725-5100	S71043	02/18	SPRINGER VERLAG	59.50			59.50	
S-5-32725-5100	S71313	03/01	AST RESEARCH, INC.	14.95			14.95	
S-5-32725-5100	S73054	05/16	CAMBRIDGE UNIVERSITY	86.00			86.00	
S-5-32725-5100	S76232	09/08	SPRINGER-VERLAG	55.00			55.00	
S-5-32725-5100	S76233	09/08	MARCEL DEKKER, INC.	35.00	67.60	32.60	0.00	COMPLETED
S-5-32725-5100	S76237	09/08	O.J. PHOTO SUPPLY	100.00	120.00	20.00	0.00	COMPLETED

APPENDIX

Papers published, accepted and submitted

1. A. Engel and F. Moss, "Mean first-passage time in random fields", Phys. Rev. A 38, 571 (1988), Rapid Communication.

2. F. Marchesoni and F. Moss, "Comment on mean first-passage time calculations for colored noise driven bistable systems", Phys. Lett. **131**, 322 (1988).

3. M. James, F. Moss, P. Hanggi and C. Van den Broeck, "Switching in the presence of colored noise: The decay of an unstable state", Phys. Rev. A 38, 4690 (1988)

4. G. Debnath, T. Zhou and F. Moss, "Comment on stochastic resonance", submitted, Phys. Rev. A.

5. K. Sinha and F. Moss, "Analog simulation of a simple system with state dependent diffusion", in press, J. Stat. Phys. (Proc. Workshop on noise and its interaction with external degrees of freedom in nonlinear dissipative systems, Los Alamos, NM, March 1988)

6. M. James, F. Moss, W. Schleich and H. Risken, "Quantum noise quenching in correlated spontaneous-emission lasers as a multiplicative noise process: The role of noise color", in revision, to be submitted, Phys. Rev. A.

TALKS GIVEN

1. "A simple system with state dependent diffusion", Workshop on noise and its interaction..., Los Alamos, NM, March 1988

3. "Holes in the probability density of strongly colored noise driven systems with bistable and random potentials", (with H. Risken) poster, Workshop on noise and its interaction..., Los Alamos, NM, March 1988

3. "Analog simulation of selected noise driven nonlinear dynamical systems", Workshop on Noisy Dynamical Systems, California Coordinating Committee for Nonlinear Studies, Lake Arrowhead, CA, May 1988.

4. "Stochastic Resonance", Physics Dept. University of Lille, France, November 1988.

5. "Stochastic Resonance", The Solvay Seminar, University of Brussels, Belgium, December 1988.

Mean first-passage time in random fields

Andreas Engel
Sektion Physik der Humboldt-Universitat, Invalidenstrasse 42, 1040 Berlin, German Democratic Republic

Frank Moss
Department of Physics, University of Missouri, St. Louis, Missouri 63121
(Received 21 March 1988)

The mean first-passage time of a particle moving in a spatially correlated, Gaussian random potential and subject to Gaussian white noise is calculated. Depending on the parameters of the problem, we find four different regimes of diffusional transport.

The mean first-passage time (MFPT) has for many years been a most useful concept in the analysis of rate processes in stochastic systems. While the origins of MFPT theory are traceable to early Russian work,[1] more recent pedagogical treatments are to be found in Stratonovich,[2] Weiss,[3] Van Kampen,[4] and Risken,[5] and today it continues as a topic for active research[6-10] and even controversy.[11] In these previous studies, various applications of the MFPT theory have been put forth, but always to bistable, metastable, or periodic potentials. In this short note, we extend the MFPT concept to include particle motion in one-dimensional random fields with nonzero correlation length and with δ-correlated random forcing: white temporal noise driving a colored spatial noise dynamics. We hope to highlight a problem in contemporary stochastic theory: temporal *colored* noise driving a spatial colored noise dynamics. Though such a problem is physically more realistic than the one presented here, the theory seems to be more difficult, especially since all temporal colored noise theories, even for completely deterministic potentials, are necessarily approximate.[6-11]

Stochastic processes in random fields represent an interesting problem in statistical mechanics. On the one hand, there is a variety of applications, for example, in disordered solid-state systems[12] or models of biological evolution.[13] On the other hand, several challenging mathematical aspects arise in their theoretical description,[14-18] and interesting applications including $1/f$ noise arise.[15,19] Previous studies mainly concentrated on the stationary probability distribution or the time dependence of the moments and the autocorrelation function. We emphasize also that in most cases previously studied, the spatial potential was δ correlated[18] in contrast to the nonzero correlated one used here. A notable exception is the early work of de Gennes.[20]

We consider the overdamped motion of a particle in a potential $U(x)$ subject to additive noise $\xi(t)$. Denoting the position of the particle by $x(t)$, the equation of motion reads

$$d_t x(t) = -d_x U(x) + \xi(t) . \tag{1}$$

Given two points x_1 and x_2, we can define for any $U(x)$ the average with respect to $\xi(t)$ of the time at which the particle for the first time reaches x_2 if it was at x_1 at $t=0$. This is the well-known MFPT (Ref. 4), $T[U(x)]$, which still functionally depends on $U(x)$. Assuming now that $U(x)$ is a sample of a random-field ensemble characterized by some functional probability density $P[U(x)]$, the question about the ensemble-averaged MFPT

$$\bar{T} = \int dU(x) P[U(x)] T[U(x)] \tag{2}$$

becomes relevant.

In the present note, we consider only the most simple case of one-dimensional motion in a Gaussian random field with Gaussian white noise, i.e., we take

$$\langle \xi(t) \rangle = 0, \ \langle \xi(t)\xi(t') \rangle = 2D\delta(t-t') ,$$

and

$$P[U(x)] = \frac{1}{N} \exp\left[-\tfrac{1}{2} \int\int dx\, dx' U(x) A(x,x') U(x') \right] , \tag{3}$$

where D measures the intensity of the temporal noise, N is a normalization constant, and $A(x,x')$ is the matrix inverse of the spatial correlation function $B(x,x')$ of the random field, i.e.,

$$\int dx' A(x,x') B(x',x'') = \delta(x-x'') .$$

Using the well-known expression for the MFPT of one-dimensional processes driven by Gaussian white noise,[4] we find, for every realization $U(x)$,

$$T[U(x)] = \frac{1}{D} \int_0^x dy \frac{1}{p^0(y)} \int_0^y dy' p^0(y') , \tag{4}$$

where

$$p^0(x) \sim \exp\left[\frac{U(x)}{D} \right] \tag{5}$$

denotes the stationary solution of the Fokker-Planck equation corresponding to Eq. (1). Here, and in the following, we assume without loss of generality that $x_1 = 0$ and $x_2 > 0$.

Equations (4) and (5) combine to give

$$T[U(x)] = \frac{1}{D} \int_0^x dy' \exp\left[-\frac{1}{D} [U(y) - U(y')] \right] . \tag{6}$$

Since $U'(y)$ occurs in (6) only linearly in the exponential, the configurational average (2), with the distribution (3), involves just a Gaussian functional integral, which is easily performed yielding

$$\bar{T}=\frac{1}{D}\int_0^x dy\int_0^y dy'\exp\left[\frac{1}{2D^2}[B(y',y')-B(y',y)-B(y,y')+B(y,y)]\right]. \tag{7}$$

For macroscopically homogeneous random fields, we have $B(x,x')=B(|x-x'|)$, and defining $B=B(0)$ we get

$$\bar{T}=\frac{1}{D}\int_0^x dy\int_0^y dy'\exp\left[\frac{1}{D^2}[B-B(y-y')]\right], \tag{8}$$

or equivalently,

$$\bar{T}=\frac{1}{D}\int_0^x dy\int_0^y dy'\exp\left[\frac{1}{D^2}[B-B(y')]\right]. \tag{9}$$

Usually, the correlation function $B(x)$ is assumed to fall off exponentially for distances large compared with the correlation length l. A typical choice is

$$B(x)=\frac{C}{l}\exp(-|x|/l), \tag{10}$$

where $C=B(l)$ defines the intensity of the static disorder. In the following, we use (10) to discuss some limiting cases of the general result as given by Eq. (8).

There are two dimensionless parameters in the problem. The first one is x/l which tells us whether there are many variations of the potential $U(x)$ between 0 and x or not. The second one is B/D^2, which represents the ratio of the intensities of spatial disorder and temporal noise. Four limiting cases are of interest as shown in Fig. 1.

We first consider the case $B/D^2 \ll 1$, i.e., the typical fluctuations of the random field are very much smaller than the intensity of the temporal noise. Expanding the exponential in (9), we find

$$\bar{T}=\frac{x^2}{2D}\left\{1+\frac{B}{D^2}\left[1-\frac{2l^2}{x^2}(e^{-x/l}-1)-\frac{2l}{x}\right]+\frac{B^2}{2D^4}\left[1-3\frac{l}{x}+\frac{l^2}{x^2}(\tfrac{1}{2}e^{-2x/l}-4e^{-x/l}+\tfrac{7}{2})+O\left[\frac{B}{D^2}\right]^3\right]\right\}. \tag{11}$$

As expected (cf. Fig. 1) this is just an expansion around the case of pure diffusion, which is characterized by the familiar Einstein result,[21]

$$\langle x^2\rangle \sim Dt. \tag{12}$$

Note that the corrections depend on the ratio x/l. For $x \ll l$, we get from Eq. (11)

$$\bar{T}=\frac{x^2}{2D}\left\{1+\frac{B}{D^2}\left[\frac{x}{3l}-\frac{x^2}{12l}\right]+\frac{B^2}{D^4}\frac{x^2}{2l^2}+O\left[\left[\frac{x}{l}\right]^3\left[\frac{B}{D^2}\right]^3\right]\right\}, \tag{13}$$

which describes the influence of smooth variations in the random field on $\bar{T}$. For $x \gg l$, Eq. (11) gives

$$\bar{T}=\frac{x^2}{2D}\left\{1+\frac{B}{D^2}\left[1-\frac{2l}{x}+\frac{2l^2}{x^2}\right]+\frac{B^2}{2D^4}\left[1-3\frac{l}{x}+\frac{7l^2}{2_x2}\right]+O\left[\left[\frac{B}{D^2}\right]^3 e^{-x/l}\right]\right\}, \tag{14}$$

which accounts for the corrections of $\bar{T}$ due to many but small fluctuations of the spatial randomness (see Fig. 1).

Of more interest is the case $B/D^2 \gg \omega$, where the spatial disorder dominates. For $x \ll l$, we then have $\exp\{(1/D^2)[B-B(y')]\} \simeq \exp\{By'/D^2 l\}$, and we obtain from (9),

$$\bar{T}\simeq\frac{D^3 l^2}{B^2}[\exp(Bx/D^2 l)-1-(Bx/D^2 l)].$$

For very large values of B/D^2 such that $Bx/D^2 l \gg 1$, we therefore find that

$$\bar{T}\simeq(D^3 l^2/B^2)\exp(Bx/D^2 l). \tag{15}$$

For $x \ll l$ the average variation of the random potential between 0 and x is of order Bx/l, and the exponential factor in Eq. (15) is just the averaged Arrhenius factor which controls escapes over such a hill in the presence of thermal noise of intensity D.

Finally, we consider the case $B/D^2 \gg 1$ and $x \gg l$. Now the potential shows many variations between 0 and x, and the particle will therefore encounter many hills and valleys on its way. First, we perform the y' integral in Eq.

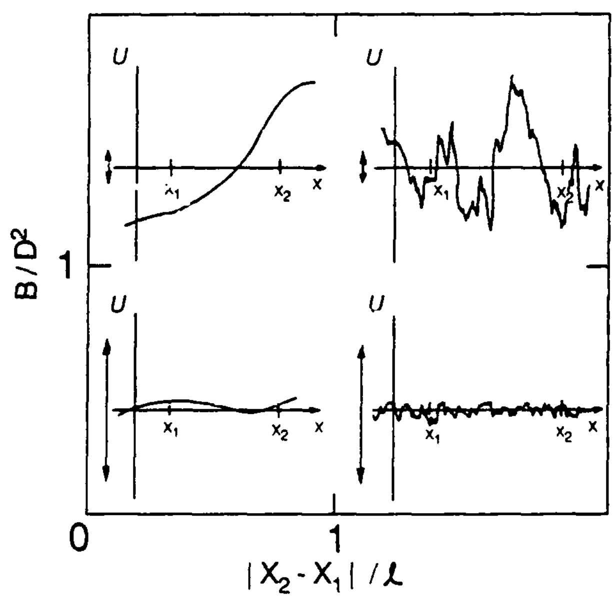


FIG. 1. Schematic plot of the random field $U(x)$ between two points x_1 and x_2 in different regions of the parameter plane. The double arrow indicates the strength of the temporal noise.

(9) exactly and obtain

$$T=(l/D)e^{B/D^2}\int_0^x dy\left[E_1\left(\frac{B}{D^2}e^{-y/l}\right)-E_1\left(\frac{B}{D^2}\right)\right],\quad(16)$$

where $E_1(x)$ is the exponential integral.[22] Because $E_1(x)\to\infty$ for $x\to 0$, the main contribution to the integral in Eq. (16) comes, for $x\to\infty$, from large values of y. Hence, we can neglect the constant term, and using[21]

$$E_1(x)=-\gamma-\ln x-\sum_{n=1}^{\infty}\frac{(-x)^n}{n\times n!}\simeq-\ln x,\ \text{for } x\to 0,\quad(17)$$

we find that

$$\bar{T}\simeq(x^2/2D)\exp(B/D^2).\quad(18)$$

As in Eq. (15), the main contribution comes from the exponential factor, which has a simple meaning. On its way from 0 to x the particle has to climb many hills (see Fig. 1), and the height $U_{\max}$ of each is a Gaussian random variable with distribution

$$P(U_{\max})\sim\exp(-U_{\max}^2/2B),\quad(19)$$

as follows from Eq. (3). The time $T(U_{\max})$ to climb such a hill is given by the Kramers formula

$$T(U_{\max})\sim\exp(U_{\max}/D).\quad(20)$$

From Eqs. (19) and (20), one realizes that the main contribution to T comes from fluctuations with height

$$U_{\max}\sim B/D.\quad(21)$$

Maxima with $U_{\max}\ll B/D$ are quickly climbed, those with $U_{\max}\gg B/D$ are very unlikely to occur. From Eqs. (20) and (21), we get $\bar{T}\sim\exp(B/D^2)$ as in (18).

The x dependence in Eq. (18) is a little surprising since the result corresponds to a process of normal diffusion with renormalized diffusion constant $D\to D\exp(-B/D^2)$. Usually, one expects that $\bar{T}$ is dominated by the transition time of the largest fluctuation between 0 and x_1. Using the distribution Eq. (19), however, one realizes that the height of this largest fluctuation varies with x as $(\ln x)^{1/2}$ which implies a contribution to $\bar{T}$ of order $\exp(\ln x)^{1/2}$. For large x, this is smaller than the pure diffusion time $T\sim x^2=\exp(2\ln x)$ and Eq. (18) is therefore explained.

Note that the problem studied in Ref. 15 is different from ours, insofar as there the derivative $d_xU(x)$ was taken as the stochastic variable. The corresponding potential $U(x)$ was therefore a Wiener process (also having zero correlation length) and hence it has statistical properties very different from ours [see Eq. (3)].

In conclusion, we have studied the ensemble-averaged MFPT for the overdamped motion of a particle in a spatially correlated Gaussian random field subject to Gaussian white noise. The general results, Eq. (7) and (9), have been shown to interpolate between several limiting cases accessible to intuitive arguments and possibly to experiment as well. For sufficiently weak spatial disorder, we found expansions around the normal diffusive behavior [Eqs. (13) and (14)], i.e., the MFPT is mainly determined by the distance to be traveled. For strong spatial disorder, the order of magnitude of the MFPT is given by an Arrhenius factor corresponding to the typical fluctuation of the random field to be climbed by the particle, but in practice, in this limit, the MFPT is determined by the local details of $U(x)$.

Discussions with Dr. L. Schimansky-Geier, Professor W. Ebeling, Professor P. Hanggi, Dr. W. Schleich, and Dr. F. Marchesoni are gratefully acknowledged. This work was supported in part by the U.S. Office of Naval Research, Grant No. N00014-88-K-0084.

---

[1] L. Pontryagin, A. Andronov, and A. Vitt, Zh. Eksp. Teor. Fiz. 3, 165 (1933), translated and reprinted in *Noise in Nonlinear Dynamical Systems, Vol. 1*, edited by F. Moss and P. V. E. McClintock (Cambridge Univ. Press, Cambridge, in press), p. 329.

[2] R. L. Stratonovich, *Topics in the Theory of Random Noise, Vol. 1* (Gordon and Breach, New York, 1963), p. 79.

[3] G. H. Weiss, Adv. Chem. Phys. **13**, 1 (1969).

[4] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981), p. 174.

[5] H. Risen, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984), p. 179.

[6] B. J. Matkowsky and Z. Schuss, SIAM J. Appl. Math. **33**, 365 (1977); C. Knessl, B. J. Matkowsky, Z. Schuss, and G. Tier, J. Stat. Phys. **42**, 169 (1986).

[7] J. M. Sancho, M. San Miguel, S. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589 (1982).

[8] P. Hanggi, T. J. Mroczkowksi, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985); P. Jung and P. Hanggi, *ibid.* **35**, 4464 (1986).

[9] R. F. Fox, Phys. Rev. A **34**, 4525 (1986); **34**, 3405 (1986).

[10] For a recent review, see K. Lindenberg, B. J. West, and J. Masoliver, in Ref. 1, Chap. 4.

[11] R. F. Fox, Phys. Rev. A **37**, 911 (1988).

[12] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981); J. Bernasconi and W. Schneider, Helv. Phys. Acta **58**, 597 (1985).

[13] W. Ebeling, A. Engel, B. Esser, and R. Feistel, J. Stat. Phys. **37**, 369 (1984).

[14] B. Derrida and Y. Pomeau, Phys. Rev. Lett. **48**, 627 (1982).

[15] E. Marinari, G. Parisi, D. Ruelle, and P. Windey, Phys. Rev. Lett. **50**, 1223 (1983); Commun. Math. Phys. **89**, 1 (1983).

[16] A. Erzan, S. Grossman, and A. Hernandez-Machado, J. Phys. A **20**, 3913 (1987).

[17] A. Engel and W. Ebeling, Phys. Rev. Lett. **59**, 1979 (1987).

[18] S. Havlin, J. E. Kiefer, and G. H. Weiss, Phys. Rev. A **36**, 1403 (1987).

[19] B. D. Hughes, E. Montroll, and M. F. Shlesinger, J. Stat. Phys. **28**, 111 (1982).

[20] P. G. de Gennes, J. Stat. Phys. **12**, 463 (1975).

[21] A. Einstein, Ann. Phys. (Leipzig) **19**, 289 (1906).

[22] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, U.S. National Bureau of Standards, NBS Circular No. 55 (U.S. GPO, Washington, DC, 1964).

COMMENT ON MEAN FIRST PASSAGE TIME CALCULATIONS FOR COLORED NOISE DRIVEN BISTABLE SYSTEMS

F. MARCHESONI
Dipartimento di Fisica, Università degli Studi e CISM, Perugia I-06100, Italy

and

Frank MOSS
Department of Physics, University of Missouri at St. Louis, St. Louis, MO 63121, USA

Received 2 March 1988; revised manuscript received 3 June 1988; accepted for publication 21 June 1988
Communicated by A.R. Bishop

Measurements of the two-dimensional probability density using an analogue simulator of a system with a "soft" bistable potential show a new, noise induced topological feature: the appearance at large noise correlation time of a symmetric pair of off-axis saddle points. The implication of this observation for mean first passage time calculations is discussed.

The problem of the rate of escape in bistable potentials driven by colored noise has recently attracted the attention of a number of investigators (for a collection of reviews see ref. [1]). Let us consider a one-dimensional bistable potential $V(x)$, with barrier located at $x=0$. In the Smoluchowski limit the stochastic relaxation of the spatial variable is described by the following equation of motion:

$$\dot{x} = -V'(x) + \epsilon(t) . \qquad (1)$$

The simplest choice for the fluctuating force $\epsilon(t)$ is given by a gaussian noise of zero-mean value with autocorrelation function (acf)

$$\langle \epsilon(t)\epsilon(0)\rangle = (D/\tau)\exp(-|t|/\tau) . \qquad (2)$$

In view of Doob's theorem [2], eqs. (1) and (2) can be replaced by a markovian system of two stochastic equations:

$$\dot{x} = -V'(x) + \epsilon ,$$

$$\dot{\epsilon} = (1/\tau)[-\epsilon(t) + \xi(t)] , \qquad (3)$$

where $\xi(t)$ denotes a zero-mean valued, gaussian, white noise with acf $\langle\xi(t)\xi(0)\rangle = 2D\delta(t)$. Two properties of system (3) make the analytical determination of the rate of escape an arduous task: (i) no manifest energy detailed balance holds [3]; and (ii) the exact stationary distributions $p(x)$ and $p(x, \epsilon)$ are unknown.

The very definition of the rate of escape may give rise to some misunderstanding. The authors of ref. [4], for instance, identify the rate of escape in bistable potentials in the small noise limit ($D \ll \Delta V$) with the smallest non-vanishing eigenvalue $\lambda(\tau)$ of the corresponding Fokker–Planck equation. This definition is particularly well suited to computations by means of numerical algorithms based on continued fraction expansions [5]. In the case of analogue or digital simulations $\lambda(\tau)$ can be determined uniquely as the reciprocal of the slowest decay-time of the temporal acf of $x(t)$ [4].

By contrast, much effort has been put [6–10] into characterizing the rate of escape in bistable potentials as the reciprocal of the mean first passage time [6] (MFPT), $T(\tau)$, for the brownian particle to cross over the potential barrier. In the white-noise limit ($\tau=0$) it can be proven that the $T^{-1}(0)$ does indeed coincide with $\lambda(0)$ up to the first order in the small parameter $D/\Delta V$ – compare the results of refs. [11]. However, the equivalence of $T^{-1}(\tau)$ and $\lambda(\tau)$

0375-9601/88/$ 03.50 © Elsevier Science Publishers B.V. (North-Holland Physics Publishing Division)

is suggested not to hold in the presence of colored noise ($\tau>0$) as indicated by the conflicting predictions for the τ-dependence of the rate of escape in refs. [4,7–10].

MFPT calculations invariably assume that for $D \ll \Delta V$, the escape from one potential well into the other occurs any time x changes sign. In the two-dimensional space (x, ϵ) this means that the barrier crossing occurs any time a single trajectory intersects the line $x=0$, the most probable crossing point being located at (0, 0). This assumption has been criticized first in ref. [10]. Subsequently, the authors of ref. [4] determined the separatrix numerically, that is, the curve of constant probability which passes through the saddle point between the peaks of the distribution function $p(x, \epsilon)$. Since $p(x, \epsilon)$ cannot be factored into a simple product of functions of ϵ and x, it is nontrivial to determine the separatrix for intermediate to large values of τ. The usual answer to such a criticism is that in the limit of very small values of τ [12] (but how small is hard to say) the correlation of x with ϵ would play a minor role: so that the most probable escape path crosses the potential barrier at the unstable point (0, 0) as in the white-noise limit. The effect of time-correlation is then supposed only to modify the curvature of the effective potential at the bottom of the well and the top of the barrier. Consequently, the corrections to the rate of escape at small τ are not expected to depend much on D, at least for small D.

In this Letter we present a result of an analogue simulation experiment, which illustrates the importance of the (x, ϵ)-space structure for a full understanding of the relaxation dynamics of the class of systems described by eqs. (1) and (2). We chose an unconventional potential function $V(x)$,

$$V(x) = -\ln\left(\frac{\mathrm{ch}\, x}{\mathrm{ch}^2 x + \mathrm{sh}^2 R}\right) + \mathrm{const}\,, \tag{4}$$

which is motivated by the analysis of the double sine-Gordon kink [13,14]. This is not the potential used in the works cited in refs. [7–10], however, it has the great advantage that certain exact results can be obtained by means of supersymmetric transformations [14]. This potential is shown in fig. 1 for $R=2$. Numerical values of the relevant potential parameters are: the barrier height,

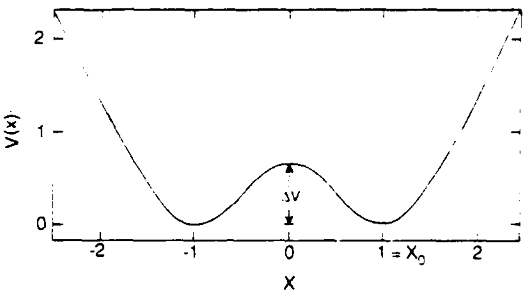


Fig. 1. The soft potential given by eq. (4).

$$\Delta V = \ln\left(\frac{\mathrm{ch}^2 R}{2\,\mathrm{sh}\, R}\right) \approx 0.67\,,$$

the positions $\pm x_0$ of the local minima,

$$x_0 = \tfrac{1}{2}\, \mathrm{ch}^{-1}\, \mathrm{sh}\, R \approx 0.98\,,$$

the curvature of the bottoms of the wells, $V''(\pm x_0) = 1 - 1/\mathrm{sh}^2 R \approx 0.92$; and finally, the curvature at the barrier, $|V''(0)| = 1 - 2/\mathrm{ch}^2 R \approx 0.86$. A unique feature of this family of potentials is that they are linear for large $|x|$, i.e., $V(x) \sim |x|$ for $x \to \pm\infty$. A good determination of $\lambda(0)$ has been obtained by means of a supersymmetric transformation [14] and the identity $T^{-1}(0) = \lambda(0)$ has been established for this potential. In addition, the slow divergence of $V(x)$ for large $|x|$ required the introduction of a modified MFPT formula [15].

We have constructed an analogue simulator of eqs. (1)–(4) using an already well established technique for simulating arbitrary potential functions [16]. Measurements of the two-dimensional statistical density $p(x, \epsilon)$ for several values of τ are shown as contour plots in fig. 2.

The main features detected by Risken and coworkers [4,5] and observed in previous simulations, both for the quartic double-well potential [17] and a sinusoidal potential [16], are reproduced here: (i) the skewing of $p(x, \epsilon)$ which increases with τ, and (ii) the separatrix which does not coincide with the axis at $x=0$ [17].

Most notably, however, a novel feature of $p(x, \epsilon)$ shows up for values of τ which are large compared with $|V''(0)| \sim V''(x_0) \sim 0.9$. For τ large enough (the magnitude increases weakly with D) two symmetric saddlepoints in the distribution located at (x_s, ϵ_s) and $(-x_s, -\epsilon_s)$ become observable with our appa-

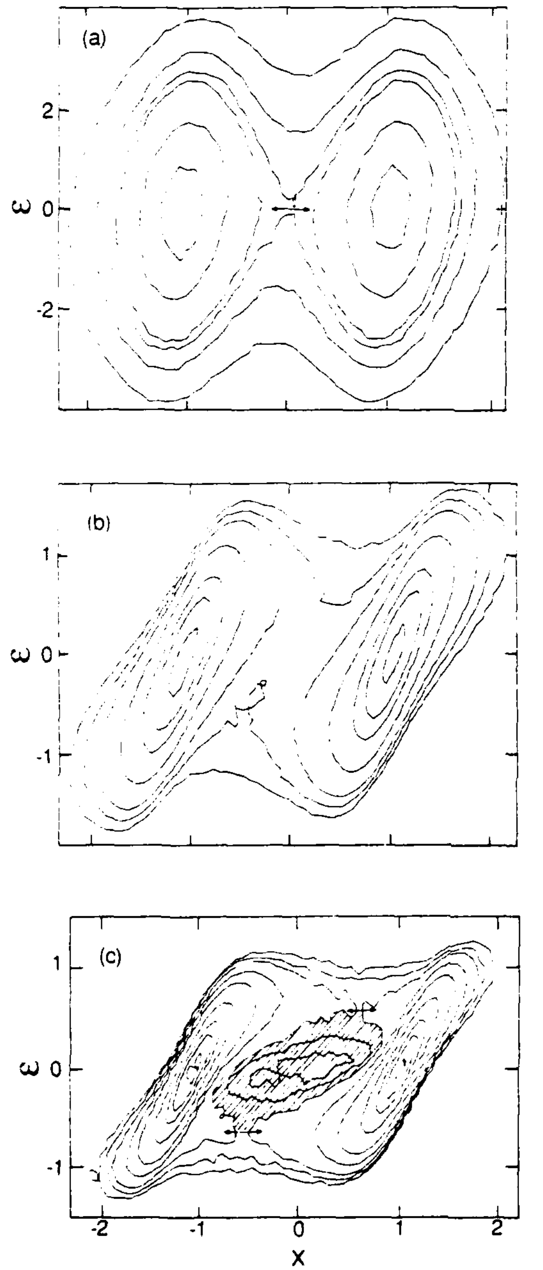


Fig. 2. Measured statistical densities $p(x, \epsilon)$ for $D=0.4$: (a) $\tau=0.1$, or quasi-white noise, with amplitudes 9700, 3400 and 1100 at the peaks, separatrix and lowest contour respectively. (b) $\tau=1.5$ where the crater has disappeared. The peak, separatrix and lowest contour amplitudes are 54700, 470, and 430 respectively. (c) $\tau=5.0$ showing the crater (shaded area) and the off-axis pair of saddle points (double arrows). The amplitudes of the peaks, saddles and lowest contours in the bottom of the crater are 98700, 110 and 20 respectively. The amplitudes are in arbitrary units.

ratus. This means that for large enough τ, the system presents two symmetric, most probable escape paths connecting the potential wells, which make a detour around the single crossing point located at (0, 0) for $\tau=0$, and run instead across the saddle points.

It is not yet clear whether this observed noise induced change in the topology of $p(x, \epsilon)$ occurs at a specific threshold value τ_{th}, or whether the two saddle points instead move out from (0, 0) continuously for $\tau > 0$.

An example is shown by the contour plot in fig. 2c for $\tau=5$, where the shaded area surrounding (0, 0) depicts the crater in $p(x, \epsilon)$. The two saddle points are marked by the double arrows, which lie on the measured contour closest to the separatrix, and are located approximately at $x_s \sim x_0/2$ and $\epsilon_s \sim \frac{1}{2}$. For comparison, densities at $\tau=1.5$, where the saddle points have just disappeared, and at $\tau=0.1$, which is close to white noise are shown in figs. 2b and 2a respectively.

A simple, physical explanation for this observation is the following: the locations of the wells of the potential under study can be approximated by $|x \pm x_0|$ (apart from an additive constant). The presence of a constant external force, for example, of magnitude $\sim\epsilon$ would correspond to a momentary tilting of the potential due to the addition of the extra term $x\epsilon$. In the neighborhood of $\epsilon \equiv \epsilon_s \approx \pm 1$, the effective potential $V(x)+x\epsilon$ becomes unstable and the trajectory falls into the single potential well. This mechanism obviously applies only when τ is large enough to allow the trajectory to relax into the new potential well before ϵ changes again. If this picture is appropriate, the noise correlation time induced change of topology is a general effect which should be observable for an arbitrary pair of wells separated by a barrier.

It is clear that the effective activation energy corresponding to the newly observed crossing points (x_s, ϵ_s) and $(-x_s, -\epsilon_s)$ is different (though perhaps not much different in this example) from the Arrhenius factors modified for use at large values of τ, for example by the ansatz of Hanggi et al. as defined in ref. [8]. We conclude by pointing out that any theory for colored noise driven bistable potentials should be able to reprodu e the complexity of the stochastic dynamics in the two variable space (x, ϵ).

We are grateful to W. Schleich for an interesting discussion and to Hans Risken for a recent discussion wherein he informed us that the off axis saddle points have very recently been observed at large correlation times for systems with periodic potentials using the matrix continued fraction technique. This work was supported in part by the U.S. Office of Naval Research, grant No. N00014-88-K-0084.

References

[1] F. Moss and P.V.E. McClintock, eds., Noise in nonlinear dynamical systems, Vol. 1 (Cambridge, 1988), to be published.

[2] W. Horsthemke and R. Lefever, Noise induced transitions (Springer, Berlin, 1984) ch. 8.

[3] F. Marchesoni, Phys. Lett. A 101 (1986) 11.

[4] P. Jung and H. Risken, Z. Phys. B 61 (1985) 367;
Th. Leiber, F. Marchesoni and H. Risken, Phys. Rev. Lett. 59 (1987) 1381;
F. Marchesoni, Phys. Rev. A 36 (1987) 4050.

[5] H. Risken, The Fokker–Planck equation (Springer, Berlin, 1986);
F. Marchesoni and P. Grigolini, J. Chem. Phys. 78 (1983) 6287.

[6] R.L. Stratonovich, Topics in the theory of random noise, Vol. 1 (Gordon and Breach, New York, 1963) p. 98.

[7] P. Hanggi, F. Marchesoni and P. Grigolini, Z. Phys. B 56 (1984) 333.

[8] P. Hanggi, T.J. Mrockowski, F. Moss and P.V.E. McClintock, Phys. Rev. A 32 (1985) 695.

[9] M. Sancho, F. Sagues and M. San Miguel, Phys. Rev. A 33 (1986) 3399;
R.F. Fox, Phys. Rev. A 33 (1986) 467, 34 (1986) 525; 3 (1988) 911;
J. Masoliver, K. Lindenberg and B.J. West, Phys. Rev. A 34 (1986) 235; 35 (1987) 3086;
C.R. Doering, P.S. Hagan and C.D. Levermore, Phys. Rev. Lett. 59 (1987) 2129;
M.M. Dygas, B.J. Matkowsky and Z. Schuss, SIAM J. Appl. Math., to be published.
P. Jung and P. Hanggi, Phys. Rev. A 35 (1987) 4464; and preprint;
K. Lindenberg, B.J. West and J. Masoliver, in: Noise in nonlinear dynamical systems, Vol. 1, eds. F. Moss and P.V.E. McClintock (Cambridge, 1988) ch. 4, to be published.

[10] P. Hanggi and P. Talkner, Phys. Rev. A 32 (1985) 1934.

[11] R.S. Larson and M.D. Kostin, J. Chem. Phys. 69 (1978) 4821;
O. Edholm and O. Leimar, Physica A 98 (1979) 313.

[12] J.M. Sancho, M. San Miguel, S.L. Katz and J.D. Gunton, Phys. Rev. A 26 (1982) 1589.

[13] D.K. Campbell, M. Peyrard and P. Sodano, Physica D 19 (1986) 165;
P. Sodano, M. El-Batanouny and C.R. Willis, Phys. Rev. B 34 (1986) 4936.

[14] M. Bernstein and L.S. Brown, Phys. Rev. Lett. 52 (1984) 1933;
R. Dutt, A. Khare and U.P. Sukhatme, Am. J. Phys. 56 (1988) 163.

[15] F. Marchesoni, P. Sodano and M. Zanetti, MIT, CTP No. 1540 (1987).

[16] K. Vogel, H. Risken, W. Schleich, M. James, F. Moss and P.V.E. McClintock, Phys. Rev. A 35 (1987) 463.

[17] F. Moss and P.V.E. McClintock, Z. Phys. B 61 (1985) 381.

Switching in the presence of colored noise: The decay of an unstable state

Mark James*
Department of Physics, University of Lancaster, Lancaster LA1 4YB, United Kingdom

Frank Moss
Department of Physics, University of Missouri at St. Louis, St. Louis, Missouri 63121

Peter Hänggi
Institute of Mathematics, University of Augsburg, D-8900 Augsburg, Federal Republic of Germany

Christian Van den Broeck
Limburgs Universitair Centrum, B-3610 Diepenbeek, Belgium
(Received 14 March 1988; revised manuscript received 29 June 1988)

Switching events are studied by means of a parametrically operated, fast transition from a monostable to a bistable potential in the continuous presence of colored noise. The problem is thus the decay of an unstable state with random initial conditions. We calculate, using contemporary colored-noise theory, and measure by analog simulation, the relaxation time to cross a reference boundary, and we contrast this with the strictly defined mean first passage time.

I. INTRODUCTION

In this paper we study a switching process under the influence of noise which is common to a class of parametrically activated bistable systems. The system exhibits a single, or monomodal, state x_0 while inactive, but develops a bimodal potential upon receipt of a parametric switching signal. The bimodal potential is developed in a time which is short compared to the characteristic dynamical response time of the system. The initial state x_0 thus becomes an unstable state at the instant when the switching signal is received. This state is, however, perturbed by noise which drives the decay. After a relaxation time $\langle T \rangle$, which we analyze and measure herein, the system settles into one of the bistable states.

Such generic switching processes were first proposed by Landauer[1] as possible zero- or low-energy switches. They have been used for quite some time as examples in discussions on the minimum energy dissipation necessary for measurement and for information transmission and computation.[2] In view of this interest and of the fundamental importance of the switching process itself, it is relevant to study the dynamics of switching in the presence of noise, which is inescapable in macroscopic systems.

The first theory and measurements on such noisy switching events induced by linearly swept parameters and their remarkable noise-averaging properties were carried out by Kondepudi *et al.*[3] Noisy switching events in lasers have been studied experimentally by Zhu *et al.*[4(a)] and analytically by Broggi *et al.*[4(b)] and later simulated with analog circuits.[5] However, all of these studies were of switching events induced by variations of the parameters on time scales comparable to the characteristic dynamical response time of the system. By contrast in the present study the switch parameter is operated by a step function at time $t=0$, thus preparing an initial state which later decays into the induced bistable states.

The problem of the decay of an unstable state in the presence of *white* noise has been well studied by a variety of techniques.[6,7] Among the more familiar and useful methods for describing the onset of macroscopic order are the time evolution of the variance $\langle x^2\rangle(t)$, the mean first passage time (MFPT), and the onset of bimodality in the probability density. The MFPT in recent years has become a widely used tool for analyzing such stochastic processes.[8]

Non-numerical colored noise theories are, however, necessarily approximate. This arises because of the extra variable necessary to describe noise with nonzero correlation time. The Langevin equation and its analogous Fokker-Planck (FP) equation are, consequently, at least two dimensional. Since the latter equation can be solved exactly only in one dimension,[9] recent years have witnessed a veritable explosion of approximative schemes,[10-17] all of which seek to reduce the FP equation to an "equivalent" one-dimensional form, and an increasingly vigorous debate regarding their accuracy and applicability.[14,17-19] With the exception of Refs. 12 and 14 all of these are adaptations of, or improvements on, an approximation originally put forth by Stratonovich[20,21] based on expansions valid for small correlation times. How accurate they are for a given correlation time depends on the application and often the noise intensity as well. Here we present measurements for τ as large as $\tau=5$ for which both one- and two-dimensional stationary probability densities have been studied.[9,12,22]

In this work the nonsmall noise color introduces difficulties at two levels. In order to calculate the stationary mean-square fluctuation intensity in the initial state, we will use a simple linearization which is expected to be accurate for small noise intensities. More detailed theories such as those cited above could be used instead, but the linear theory is sufficiently accurate for the

initial-state calculations. We then analyze the transient relaxation toward the stable state using colored-noise modified Suzuki scaling[6] following a recent theory.[23] These results are then compared to measurements made on an analog simulator of an example switch.

We find that the relaxation time is a decreasing logarithmic function of the noise intensity, results that, *for white noise,* were anticipated much earlier.[6,24–31] Increasing the noise correlation time increases the relaxation time but does not have a large effect.

This paper is organized as follows. In Sec. II the theory is developed following Refs. 12 and 23. In Sec. III the simulator, its operation, and the measurement techniques are described. The results are presented in Sec. IV and compared to the calculations. Conclusions and a discussion are presented in Sec. V.

II. THEORY

First we consider the stationary dynamics in a single-well potential, i.e., before the control-parameter-induced switching event. The potential is given by

$$U(x)=\tfrac{1}{2}[x^2-A\ln(1+x^2)]\ , \tag{1a}$$

and the forcing is

$$f(x)=x[-1+A(t)/(1+x^2)]\ , \tag{1b}$$

where $A(t)$ is the control parameter. As shown in Fig. 1, the potential is monostable for $A<1$ and bistable for $A>1$. Moreover, the width of the monostable potential depends upon $|A|$. Our model is determined by the Langevin equation

$$\dot{x}=f(x)+\xi(t)\ , \tag{2a}$$

where $\xi(t)$ is an exponentially correlated, Gaussian noise source with zero mean:

$$\langle \xi(t)\xi(s)\rangle=(D/\tau)\exp(-|t-s|/\tau)\ , \tag{2b}$$

where D is the noise intensity and τ is the noise correlation time. The system is prepared in the monostable initial state by setting $A(t)=A_0<1$, for which the deterministic solution $x_0=0$, is globally stable. After having been prepared in this initial state for a time sufficiently long to establish stationary statistical conditions, the system is switched at $t=0$ so that $A(t)=A>1$ for all $t>0$. The control parameter is thus a step function.

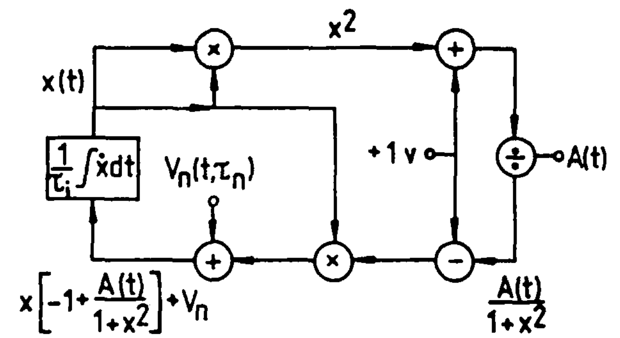


FIG. 2. The circuit diagram of the electronic simulator. The multipliers and the divider are standard chips available from Analog Devices, Norwood, Mass.

We first consider the initial state for which stationary conditions are assumed. In order to proceed it is necessary to obtain the stationary probability density. In fact, we will only need the second moment $\langle x^2\rangle_{st}=\langle x_0^2(\tau)\rangle$ of the stationary initial distribution. Since both the experiment and the theory to be used later are restricted to small values of D, linearized theory will be sufficient for calculating $\langle x_0^2(\tau)\rangle$. We begin with the following FP equation, valid for small D:

$$\frac{\partial P(x,\xi,t)}{\partial t}=\frac{\partial}{\partial x}\{[(|A_0|+1)x-\xi]P(x,\xi,t)\}+\frac{\partial}{\partial \xi}\left[\frac{\xi}{\tau}P(x,\xi,t)\right]+\frac{D}{\tau^2}\frac{\partial^2}{\partial \xi^2}P(x,\xi,t)\ . \tag{3}$$

In the steady state, the equations for the moments read

$$(|A_0|+1)\langle x_0^2(\tau)\rangle-\langle x\xi\rangle=0\ , \tag{4}$$

$$(|A_0|+1+\tau^{-1})\langle x\xi\rangle-\langle \xi^2\rangle=0\ , \tag{5}$$

$$-\langle \xi^2\rangle+\frac{D}{\tau}=0\ , \tag{6}$$

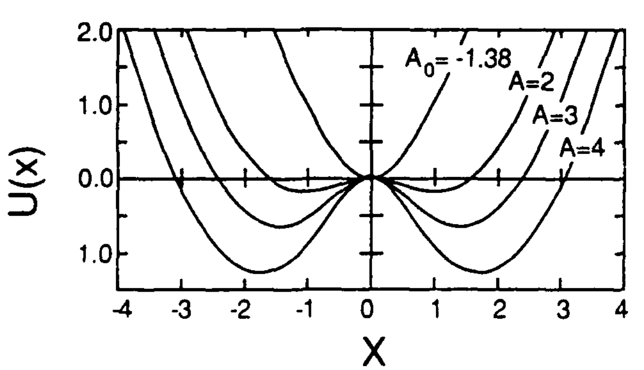


FIG. 1. The potential, showing a monostable initial state for $A_0=-1.38$ and the switched bistable states at the values of A indicated.

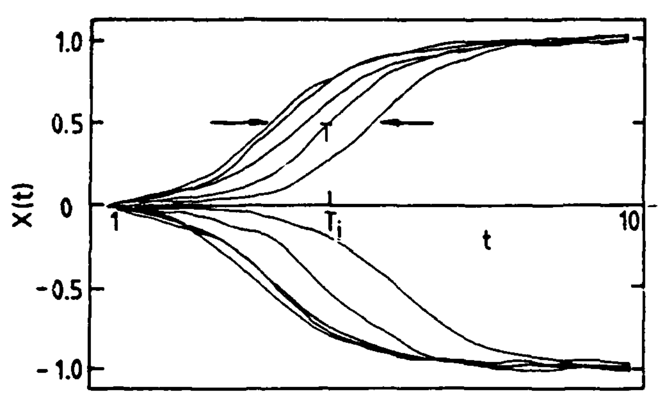


FIG. 3. Example trajectories measured for $A_0=-1.38$ and $A=2$ for $D=0.1$ and $\tau=1.0$. The bistable potential is switched at $t=0$. The vertical scale is in volts with the potential minima at ± 1.0 V. The horizontal scale is in ms.

FIG. 4. Examples of the probability density of the relaxation time from which the $\langle T \rangle$ were obtained for $\tau=1.0$, $A=2.0$, $A_0=-1.38$, and, reading from left to right, $D_1=11.7\times10^{-3}$ V^2, $\langle T \rangle=2.4$ ms; $D_2=0.20\times10^{-3}$ V^2, $\langle T \rangle=4.4$ ms; and $D_3=2.56\times10^{-6}$ V^2, $\langle T \rangle=6.6$. The horizontal scale is in ms.

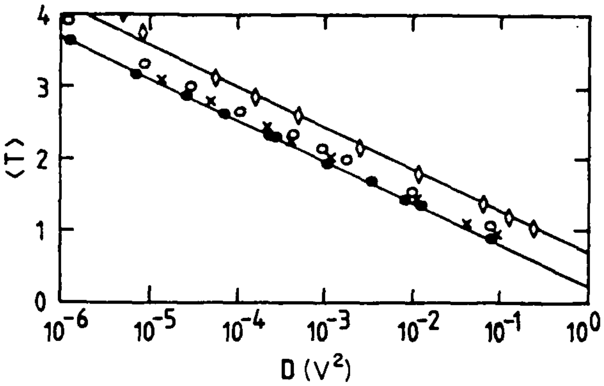


FIG. 6. $\langle T \rangle$ in ms vs D in V^2 for $A_0=-1.38$ and $A=3.0$. The values of τ are 0.1, ●; 0.5, ×; 1.0, ○; and 4.9, ◊. The solid lines are fits to the data.

from which we conclude that

$$\langle x_0^2(\tau)\rangle = D\{(1+|A_0|)[1+\tau(1+|A_0|]\}^{-1}+O(D^2)\ . \tag{7}$$

This describes the stationary dynamics before the switching event takes place.

For $t=0^+$, the system is switched to a bistable state as shown in Fig. 1 for $A>0$, and the state at $x=0$ is rendered locally unstable. Since the switching event takes place in a time very short compared to all other time scales in the problem, the decay is driven by the noise dynamics described by Eqs. (6) and (7) toward the two locally stable states $x_s=\pm\sqrt{A-1}$ created at $t=0$.

The simulation, described in Sec. III, measures the residence time T, for a random walker $x(t)$ necessary to cross a reference value $x_R=x_s/2$ for the first time after the switch function is activated at $t=0$. The mean of this quantity $\langle T \rangle$ is a measure of the time scale on which the system assumes macroscopic order and is closely related to (but not identical with) the MFPT on the same interval. The residence time is like a weighted MFPT wherein the distribution of initial conditions is a Gaussian whose second moment is given by Eq. (7). By contrast, our switch is more physically realistic in supposing that the noise is continuously present, as it is in all real switches. In fact it is possible that the instant the unstable state is born, the trajectory $x_0(t=0)$ could already exceed the reference boundry x_R. For small D, however, in the range where both the theory and the simulation are accurate, such events are extremely rare.

The time for decay from an unstable state has been studied by many authors.[6,26–31] As shown first by Kubo *et al.*,[26] the relaxation time exhibits a logarithmic dependence on the noise intensity. Very recently, Suzuki's scaling theory[6,31] has been generalized to include colored noise,[23] with the result

$$\langle T \rangle = -(1/2\alpha)\ln(C\{\langle x_0^2\rangle + D[\alpha(1+\alpha\tau)]^{-1}\})\ , \tag{8}$$

where $\alpha\equiv f'(x=0^+)=A-1$, and where C is a constant that depends only on the parameters of the deterministic nonlinear flow $f(x)$, i.e., on A_0 and A, but not on D or τ. The theoretical value of C depends also on the detailed definition of the residence time, for example, as the time for the second moment $\langle x^2\rangle(t)$ to relax to some reference value x_R, or the time required for $P(x,t)$ to become bimodal, or the strictly defined MFPT, etc. But certainly C is of order unity so that $\ln C$ is of order zero.

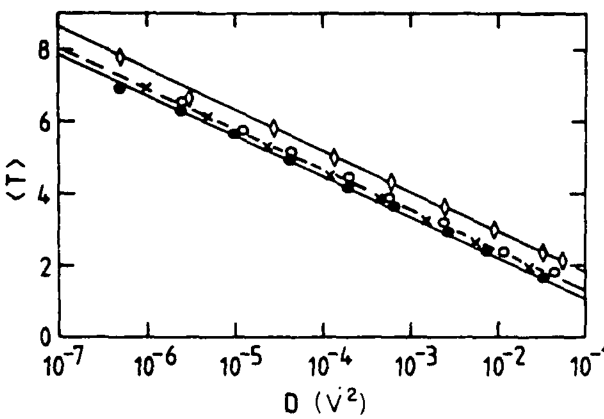


FIG. 5. $\langle T \rangle$ in ms vs D in V^2 for $A_0=-1.38$ and $A=2.0$. The values of τ are 0.1, ●; 0.5, ×; 1.0, ○; and 4.9, ◊. The solid lines are fits to the data and the dashed line is a predicted result for $\tau=0.5$.

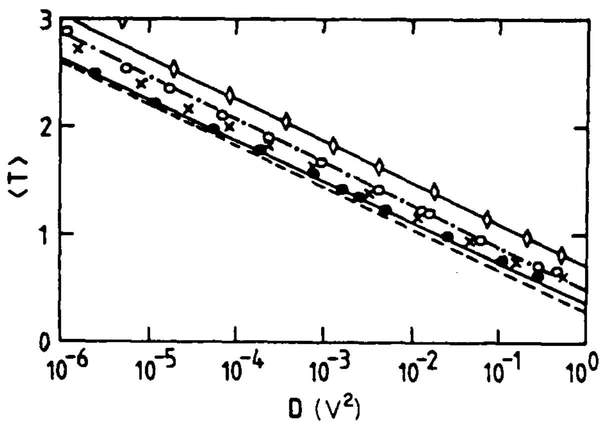


FIG. 7. $\langle T \rangle$ in ms vs D in V^2 for $A_0=-2.40$ and $A=4.0$. The values of τ are 0.1, ●; 0.5, ×; 1.0, ○; and 5.0, ◊. The solid lines are fits to the data. The dot-dashed and dashed lines are theoretical predictions for $\tau=1.0$ and 0.1, respectively.

TABLE I. Calculated and measured parameters for $A_0 = -1.38$, $A = 2$, and $B_0 = -0.03$.

τ	m^{theor}	m^{expt}	$b^{\text{theor}}(\tau)$	$b^{\text{theor}}(\tau)+B_0$	b^{expt}
4.9	1.15	1.11	0.80	0.77	0.77
1.0	1.15	1.14	0.23	0.20	0.20
0.5	1.15	1.13	0.08	0.05	0.10
0.1	1.15	1.12	−0.11	−0.14	0.02

In our experiment, the noise color enters Eq. (8) in two places: First, there is a colored-noise effect on the initial state $P_0(x,\tau)$ which leads to $\langle x_0^2(\tau)\rangle$ as given by Eq. (7). Second, the dynamics of the decay is dependent on τ as shown explicitly in Eq. (8). Both effects move $\langle T\rangle$ in the same direction, i.e., increasing τ results in increasing $\langle T\rangle$.

Observing that $\ln x = \ln(10)\log_{10}x$, we can recast these results in the convenient form

$$\langle T\rangle = -m\,\log_{10}D + b(\tau) + B_0\ , \tag{9}$$

where

$$m = \frac{\ln 10}{2(A-1)}\ , \tag{10a}$$

and

$$B_0 = -m\,\log_{10}C \sim 0\ , \tag{10b}$$

so that $b(\tau)$ describes the combined dependence of $\langle T\rangle$ on the noise correlation time:

$$b(\tau) = -m\,\log_{10}\{(\,|A_0|+1)^{-1}[1+\tau(\,|A_0|+1)]^{-1} + (A-1)^{-1}[1+\tau(A-1)]^{-1}\}\ . \tag{11}$$

These results can be accurate only in the range of small D. Indeed, we used linear theory to calculate $\langle x_0^2\rangle$. Furthermore, Suzuki scaling is expected to be accurate only for small D, but here the quantitative limit is not known.

In the following sections we describe an analog simulation of Eqs. (1) and (2) for the range $0.1 \le \tau \le 5$ and for $10^{-6} \le D \le 10^{-1}$. Since $D = \tau\langle\xi^2\rangle$, as shown by Eq. (2b), the range of D corresponds to a range of noise voltage $V_n \equiv \xi$ of somewhat less than three orders of magnitude, or about 10 mV to a few volts, which corresponds to the usable dynamic range of our analog simulators.

III. ANALOG SIMULATION

We have constructed a circuit model of Eqs. (1) and (2) using by now quite standard techniques.[32] The schematic diagram of this simulator is shown on Fig. 2.

The noise voltage $V_n(t)$, is obtained from a Wandell and Goltermann[33] wide band (>100 kHz), Gaussian noise generator. In order to create colored noise, $V_n(t)$ is passed through a linear, single pole filter with a transfer function $H(\omega) = [1+(\omega\tau_n)^2]^{-1}$, where ω is the radian frequency and τ_n the noise correlation time. The characteristic response time of the simulator is established by the integrator time constant τ_i, as shown on Fig. 2. The dimensionless noise correlation time, as it appears in the theory and in Eq. (2), is the ratio $\tau = \tau_n/\tau_i$. When $\tau \ll 1$, the simulator perceives the noise as quasiwhite, whereas $\tau \ge 1$ marks the colored-noise regime. With $V_n(t) \equiv \xi(t)$, Eq. (2b) defines the noise intensity $D = \tau\langle V_n^2\rangle$ for $t = s$. The mean-square noise voltage $\langle V_n^2\rangle$ is the measured quantity in our simulation which defines D. In this experiment we always maintained $D \le 0.30$, and the correlation time varied over the range $0.1 \le \tau \le 5$.

In the absence of noise, the discrepancies between the deterministic steady states measured on the simulator are smaller than $\pm 2.5\%$ when compared to the steady-state ($\dot{x} = 0$) solutions of Eqs. (1) and (2a). Measurements in the presence of noise are, of course, subject to statistical errors which can be reduced by increasing the number of samples in a given average. In this experiment the statistical errors are estimated (from the repeatability) to be $\simeq \pm 5\%$. The largest error, and the most difficult to quantify, is systematic and shows up in the quasiwhite noise end of the range of τ. This results from the limited dynamic range and bandwidth of the simulator. For τ small, $V_n(t)$ is large, and an increasing number of its large-amplitude excursions in the wings of the Gaussian are clipped as τ is decreased. At $\tau = 0.1$, this results in discrepancies between our measured probability densities and white-noise solutions of the FP equation which, in places, are as large as $\sim 20\%$. In this experiment, relaxation times are measured for which it is difficult to estimate the systematic errors at the small-τ end.

In operation, a rectangular wave $A(t)$, operating between the voltages A_0 and A was applied to the divider as shown in Fig. 2. The frequency of this wave was adjusted so that in the state A_0 sufficient time was allowed for the initial probability density of x to become stationary (several hundred times τ_i). The wave switches from A_0 to A at $t = 0$ and at the same time the data-analysis

TABLE II. Calculated and measured parameters for $A_0 = -1.38$, $A = 3$, and $B_0 = 0.07$.

τ	m^{theor}	m^{expt}	$b^{\text{theor}}(\tau)$	$b^{\text{theor}}(\tau)+B_0$	b^{expt}
4.9	0.58	0.58	0.63	0.70	0.70
1.0	0.58	0.58	0.31	0.38	0.39
0.5	0.58	0.58	0.20	0.27	0.31
0.1	0.58	0.57	0.07	0.14	0.28

TABLE III. Calculated and measured parameters for $A_0=-2.4$, $A=4.0$, and $B_0=0.16$.

τ	m^{theor}	m^{expt}	$b^{\text{theor}}(\tau)$	$b^{\text{theor}}(\tau)+B_0$	b^{expt}
5.0	0.38	0.39	0.55	0.71	0.71
1.0	0.38	0.39	0.32	0.48	0.47
0.5	0.38	0.39	0.24	0.40	0.39
0.1	0.38	0.38	0.12	0.28	0.35

system, a Nicolet-Lab 80 computer and digitizer connected to the circuit at $x(t)$, was triggered. A time series of typically 2000K digitized points was then obtained, and the time at which the trajectory first crossed the threshold $x_R=x_s/2$ ($x_s=\pm\sqrt{A-1}$ are the deterministic steady states) was measured and stored. Ten example trajectories are shown in Fig. 3, where the threshold is marked by arrows, and an example crossing time at T_i is shown. After a large number of such measurements (typically 10^4) the computer tabulates the mean-relaxation time $\langle T\rangle$ and its density $P(T)$. Three examples of the densities are shown in Fig. 4 for three values of D. As expected $\langle T\rangle$ increases as D becomes smaller.

IV. RESULTS

The results of our simulation are summarized on three graphs and compared to the theoretical predictions in three tables. Figure 5 shows our measured values of $\langle T\rangle$ versus D for four values of τ as indicated by the different symbols for $A_0=-1.38$ and $A=2.0$ V. Each set of data were matched by least-squares fit to the equation

$$\langle T\rangle=-m\log_{10}D+b\ , \tag{12}$$

and the values m^{expt} and b^{expt} were extracted. These were compared to the results predicted by Eqs. (9)–(11). As shown by Eq. (9) the entire colored-noise effect is represented by the constant $b(\tau)$. It is evident, however, that the non-color-dependent constant B_0 is not negligible. We obtained a value for B_0 by matching the experimental and theoretical results at $\tau=4.9$ and 5 (where the simulation is most accurate) and then compared the predicted and observed values in Table I, where the $b^{\text{theor}}(\tau)$ and m^{theor} are calculated directly from Eqs. (10a) and (11), and b^{expt} is to be compared with $b^{\text{theor}}(\tau)+B_0$. The results are shown on Table I. The largest discrepancies are for $\tau=0.5$ and 0.1 as expected. Even so, on the logarithmic scale of Fig. 5 the difference between Eq. (9) using the theoretical values and the data are small, as shown by the dashed line which is to be compared to the crosses $\tau=0.5$). The solid lines through the data for $\tau=0.1$ and $\tau=4.9$ are example plots of the results of the least-squares fits of the data to Eq. (12).

Figure 6 and Table II show the results for $A_0=-1.38$ and $A=3.0$. It is evident that while B_0 has changed considerably, the discrepancy at $\tau=0.1$ is decreased. Figure 7 and Table III show the results for $A_0=-2.4$ and $A=4$. On Fig. 7 we also show the theoretical result for $\tau=1.0$ as the dot-dashed line to be compared with the open circles. The dashed line is the prediction for $\tau=0.1$ and shows the largest disagreement with the measurements (closed circles).

V. CONCLUSIONS AND DISCUSSION

The agreement between the calculated and the measured results in this work is satisfactory considering the approximations necessary to achieve a colored-noise theory. We emphasize further that the strictly colored-noise contributions to the theory have two sources: the decay theory of Dhara and Menon[23] and the *ansatz*.[12] The latter, which agrees with the linearization result in Eq. (7), can be used to calculate the second moment of the initial, stationary ($t<0$) density, an application for which it is known to be relatively accurate. The relative importance of each of these contributions depends on A_0 and A as shown by the two terms in brackets in Eq. (11). For small A_0 and large A, $\langle x_0^2\rangle$ dominates the decay process and hence the *ansatz* is more important, while for large A_0 and small A the reverse is true, and the nonlinear decay process dominates. We conclude by pointing out that colored-noise-driven decay of unstable states should find applications in a variety of switching scenarios most notably in laser dynamics and nonlinear optical bistability.

ACKNOWLEDGEMENTS

Two of us (F.M. and P.H.) are grateful to the Physics Departments of the University of Lancaster, U. K. and Limburg's University Centrum, Belgium for their hospitality during which part of this work was carried out. We are pleased to acknowledge stimulating discussions with F. T. Arecchi, E. Arimondo, W. Ebeling, L. A. Lugiato, P. Mandel, F. Marchesoni, L. Narducci, H. Risken, R. Roy, J. M. Sancho, M. San Miguel, and J. Tredicce. This work was supported in part by the British Science and Engineering Research Council, the Belgian National Foundation for Scientific Research, NATO Grant No. RG. 85/0770, and the U. S. Office of Naval Research Grant No. N00014-88-K-0084.

*Present address: Division of Science, University of Wisconsin at Parkside, Kenosha, WI 53141.

[1]R. W. Keyes and R. Landauer, IBM J. Res. Dev. **14**, 152 (1970); R. Landauer, Appl. Phys. Lett. **51**, 2056 (1987); in *Der Informationsbegriff in Technik and Wissenschaft*, edited by O. G. Folberth and C. Hackl (Oldenbourg, Munich, 1986), p. 139.

[2]R. Landauer, Phys. Scr. **35**, 88 (1987); C. H. Bennett, Sci. Am.

257, 108 (1987); C. H. Bennett and R. Landauer, *ibid.* **253**, 48 (1985). For a review see C. H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982).

[3]D. K. Kondepudi, F. Moss, and P. V. E. McClintock, Physica (Utrecht) **21D**, 296 (1986).

[4](a) S. Zhu, A. W. Yu, and R. Roy, Phys. Rev. A **34**, 4333 (1986); (b) G. Broggi, A. Colombo, L. A. Lugiato, and P. Mandel, *ibid.* **33**, 3635 (1986).

[5]R. Mannella, F. Moss, and P. V. E. McClintock, Phys. Rev. A **35**, 2560 (1987).

[6]For an authoritative review, see M. Suzuki, Adv. Chem. Phys. **46**, 195 (1981).

[7]P. de Pasquale and P. Tombesi, Phys. Lett. **72A**, 7 (1979).

[8]For a collection of recent reviews, see Rate Processes and First Passage Times, edited by G. H. Weiss [J. Stat. Phys. **42**, 1 (1986)].

[9]Matrix continued-fraction technique offer solutions of two-dimensional FP equations which are, in principle, exact. See H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin 1984). The necessity to invert and hence truncate a final matrix of infinite dimension limits the practical application of the technique, using ordinary computers, to two-dimensional problems. For MCF theory applied to the decay of an unstable state, see P. Jung and H. Risken, Z. Phys. B **59**, 469 (1985).

[10]P. Hanggi and P. Talkner, Phys. Rev. Lett. **51**, 2242 (1983).

[11]J. M. Sancho, M. San Miguel, S. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589 (1982).

[12]P. Hanggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985).

[13]R. F. Fox, Phys. Rev. A **33**, 467 (1986); **34**, 4525 (1986).

[14]P. Jung and P. Hanggi, Phys. Rev. A **35**, 4464 (1987).

[15]M. Dygas, B. J. Matkowsky, and Z. Schuss, SIAM J. Appl. Math. **48**, 425 (1988).

[16]C. R. Doering, P. S. Hagan, and C. D. Levermore, Phys. Rev. Lett. **59**, 2129 (1987).

[17]R. F. Fox, Phys. Rev. A **37**, 911 (1988).

[18]See the various articles in *Noise in Nonlinear Dynamical Systems*, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, 1989), Vol. I.

[19]P. Jung and P. Hänggi, Phys. Rev. Lett. **61**, 11 (1988).

[20]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963), Vol. I.

[21]For a review see, N. G. Van Kampen, Phys. Rep. **24C**, 171 (1976).

[22]F. Moss, P. Hänggi, R. Mannella, and P. V. E. McClintock, Phys. Rev. A **33**, 4459 (1986).

[23]A. K. Dhara and S. V. G. Menon, J. Stat. Phys. **46**, 743 (1987).

[24]F. T. Arecchi, A. Politi, and L. Ulivi, Nuovo Cimento **71B**, 119 (1982).

[25]G. Broggi, L. A. Lugiato, and A. Colombo, Phys. Rev. A **32**, 2803 (1985).

[26]R. Kubo, K. Matsuo, and K. Kitahara, J. Stat. Phys. **9**, 51 (1971).

[27]F. T. Arecchi, V. Degiorgio, and B. Querzola, Phys. Rev. A **3**, 1108 (1971); F. T. Arecchi and A. Politi, Phys. Lett. **45**, 1219 (1980).

[28]F. Haake, Phys. Rev. Lett. **41**, 1685 (1978).

[29]U. Weiss, Phys. Rev. A **25**, 2444 (1982).

[30]F. de Pasquale, P. Tataglia, and P. Tombesi, Phys. Rev. A **25**, 466 (1982).

[31]M. Suzuki, Prog. Theor. Phys. **56**, 77 (1976); **56**, 477 (1976); **57**, 380 (1977).

[32]For a review, see P. V. E. McClintock and F. Moss, in Ref. 18, Vol. III, Chap. 9.

[33]Wendel and Goltermann, GmbH & Co., P. O. Box 45, D7412 Eningen, Federal Republic of Germany.

Comment on Stochastic Resonance

G. Debnath*, T. Zhou+ and Frank Moss

Department of Physics

University of Missouri at St. Louis

St. Louis, MO 63121

ABSTRACT

A recent and interesting experimental paper [B. McNamara, K. Wiesenfeld and R. Roy, Phys. Rev. Lett. 60, 2626 (1988)] has refocused attention on the problem of stochastic resonance by presenting measurements of the signal-to-noise ratio (SNR) of a noise driven, periodically modulated bistable ring laser. We point out that the theoretical SNR, as defined in this and a previous work, is always infinity, because additive modulation leads to a delta function in the power spectrum of the output. Quantitative information on stochastic resonance is contained in the *strength* of this delta function relative to the noise background. We qualitatively reproduce the SNR data with an analog simulator using a standard quartic bistable potential. In this, as in previous experiments and simulations, a peak in the observed power spectrum is a reflection of the delta function, but the amplitude of the peak is rendered finite (and hence measurable) only because of the finite resolution of the measurement system.

PACS numbers: 5.40. + j, 2.50. + s, 42.65.Pc

The phenomena of stochastic resonance was first investigated by Benzi, Sutera and Vulpiani[1] and later suggested by C. Nicolis[2] and by Benzi, Parisi, Sutera and Vulpiani[3] as a possible explanation of the observed periodicity in the recurrence of the ice ages. In this model, a pair of stable climate states separated by a barrier is imagined. This bistable system is driven by noise resulting from random fluctuations of the solar constant. Using reasonable climate models, it was demonstrated that such fluctuations could trigger switching events between the two stable states on time scales which are in approximate agreement with the period of the observed recurrences (10^5 yrs)[4,5]. The switching events would, however, be uncorrelated random occurrences in time. In order to explain the observed periodicity, a modulation of the height of the barrier or of the alternate depths of the potential wells was introduced into the model. The resulting switching events, while still randomly occurring, must now be correlated with the periodic forcing since the switching probability is a strong function of the well depth. In the climate models, the modulation is assumed to result from a weak but periodic variation in the eccentricity of the earth's orbit with period $\cong 10^5$ yrs. Digital simulations of the models show finite amplitude peaks in the power spectra located at the modulation frequency[1,3].

Several years ago, a physical realization of stochastic resonance was demonstrated by Fauve and Heslot[6] using an electronic Schmidtt trigger as a bistable system. They modulated periodically the depths of the potential wells, which represent the two stable states of the switch, alternately and simultaneously applied additive white noise. The measured power spectrum of this system displayed a sharp peak at the modulation frequency superimposed on a slowly varying continuum noise background of Lorentzian line shape. These authors defined the signal-to-noise ratio (SNR) as the (measured) amplitude of the peak relative to the (measured) noise back-

ground. They observed that the SNR passed through a maximum as the noise intensity D was increased from zero, and that at the maximum the value $D = D_0$ could be associated with the period T of the modulating function by a Kramer's characteristic time (see also Ref. 2), $\tau = T \propto \exp(\Delta u/D_0)$, where ΔU is the unmodulated barrier height.

Very recently, McNamara, Wiesenfeld and Roy have observed this phenomenon using a bidirectional ring laser as the bistable system[7]. Using an intracavity acousto-optic modulator, they were able to induce changes in the direction of the lasing by controlling the acoustic frequency. Representing the two directions as the stable states separated by a barrier, it was possible to modulate the height of the barrier and to introduce external noise as well by modulating the acoustic frequency. These authors have observed the same phenomenology as those of Ref. 6. In particular, they have observed a sharp peak in the power spectrum of the laser intensity (measured in one direction) superimposed on a broadband noise background spectrum. In order to obtain the SNR, they measured the *amplitude* of the peak and that of the noise background at the modulation frequency. Measurements of the SNR versus the noise intensity in this experiment demonstrated the characteristic maximum, though the Kramer's time was not obtained.

In this Comment, we point out, based on physical arguments, that the power spectra of all such systems regardless of the details of the model, but additively modulated by a single frequency must contain delta functions. Noise in the system does not alter this so long as the noise does not multiply the amplitude of the periodic modulation. By contrast, experimental measurements of the power spectra or digital simulations of models, both obtained from Fourier transforms of time series made at finite resolution, show non-infinite amplitude peaks. The amplitudes so obtained ref-

lect *both* the strength of the singularity, which contains all the information about stochastic resonance, and the finite resolution of the Fourier transform technique used. The latter is an entirely instrumental effect which also determines the observed line shape. Results obtained by measurements of the amplitudes only of the peaks in the power spectra can therefore only be qualitative in the absence of the details of the line shape[8]. This appears to have escaped notice not only in the recent experiments[6,7] but also in the previous digital simulations[3].

For the purposes of discussion, we consider two models represented by the following Langevin equations:

$$\dot{x} = ax - x^3 + \epsilon\cos\omega t + \xi(t), \qquad \epsilon < (4a^3/27)^{1/2}, \tag{1}$$

and

$$\dot{x} = a(t)x - x^3 + \xi(t), \qquad a(t) = 2(1 + \epsilon\cos\omega t)^{1/2}, \quad \epsilon < 1, \tag{2}$$

which represent an infinitely damped system moving in the standard quartic potential,

$$U(x) = -(a/2)x^2 + (1/4)x^4, \tag{3}$$

modulated at the frequency ω and driven by the additive noise $\xi(t)$. Equation (1) represents additive modulation and additive noise. In this case, the modulation raises and lowers the depth of each well alternately on alternate half cycles. Equation (2) represents the case of multiplicative modulation and additive noise[9]. In this case, the height of the potential barrier $\Delta U = (1/4)a^2$ is modulated about an average value of 1. The limits on ϵ insure that the barrier never vanishes. (In the simulation discussed

below, most of the data were taken for $a = (2)^{1/2}$ and $\epsilon = 0.4$).

As Eckmann and Thomas have pointed out, calculations of the statistical properties of such time modulated bistable systems are by no means trivial[10], in the first instance because they are nonlinear, and in the second because they are not stationary. Nevertheless, some general remarks can be made. Considering Eq. (1), whatever the Fourier transform of x(t) may be, we note that the modulation term, $\epsilon\cos\omega t$, stands alone and therefore will necessarily contribute a delta function at ω to the transform and hence to the power spectrum. In Eq. (2), or in other versions where the modulation may multiply more highly nonlinear terms, the situation becomes more complicated because x(t) itself is a stochastic function. The Fourier transform can, in principle, be broadened into a continuum which may be repeated at harmonics of the modulation frequency. In the simulation of Eq. (2) described below, however, the measured line shapes are as sharp as or sharper than those observed in the simulation of Eq. (1). In neither case is the line shape observed to be continuously broadened either by the noise or by the dynamics (see Ref. 9). Theoretical results recently obtained by Fox predict Lorentzian line shapes even for Eq. (1)[11]. By contrast, a generalized two state model put forth by McNamara and Wiesenfeld predicts delta functions in the power spectrum.[12]

The climate models[1-3] are based on a Langevin equation of the type of Eq. (2). The Schmidtt trigger[6] obeys an equation of the type of Eq. (1). An accurate model for the laser is more difficult for two reasons. First, the laser is a multidimensional system, and even with appropriate adiabatic eliminations probably cannot be adequately represented by any less than two coupled Langevin equations. Second, the details of the bistable potential actually introduced by the AOM are not well understood[13]. Nevertheless, the modulation and the noise must enter the experimental sys-

tems phenomenologically either as additive or multiplicative terms or some combination of the two. We believe that the generic systems represented by Eqs. (1) and (2) above will thus reproduce the phenomenology observed in the experimental systems.

In order to illustrate these remarks, we have built analog simulators of Eqs. (1) and (2) following well developed techniques[14]. The modulation frequency was always set at $f = \omega/2\pi = 500$ Hz, and the modulation amplitude was always $\epsilon = 0.4$. An example measured time series x(t), obtained from the simulator of Eq. (1) is shown in Fig. 1 (a). The time scale is delineated by the plot of the 500 Hz modulation shown in Fig. 1 (b). The switching events occur randomly in time and, in this example, on a somewhat longer time scale than the modulation period. They are, however, correlated with the modulation as shown by the sharp peak at exactly 500 Hz in the measured power spectrum shown in Fig. 1 (c). Figure 1 (d) shows a power spectrum obtained from a simulator of Eq. (2). It is very similar except that a small peak appears at the second harmonic of the modulation frequency. These power spectra are qualitatively very similar to the ones published by Fauve and Heslot[6] and by McNamara, *et. al.*[7] in the sense that very sharp peaks are observed superimposed on a broadband noise background.

The power spectra were measured in the following way: First 2048 points of time domain data x(t) were digitized with 12 bit accuracy, each point separated from its neighbors by 300 μs. The magnitude of the square of the Fourier transform was then computed and compressed into 1024 points. The final power spectrum was obtained by averaging a number of samples (usually 200) of the individual spectra. In every case, the peak at the modulation frequency was only a few points wide, and the measured amplitude of the peak depended upon which individual point was chosen as "the maximum". Increasing the frequency resolution by increasing the sep-

aration time between digitized points resulted in narrower, higher amplitude peaks and the reverse was also true though we explored a range of only a factor of two in separation time.

Using the definition adopted in Refs. 6 and 7 (the ratio of the amplitudes of the peaks to the background noise power density) we have measured the SNR as a function of noise intensity $D = \tau<\xi^2>$, where τ is the noise correlation time[15], for the simulator of Eq. (1). The data are shown by the experimental points in Fig. 2 and are self consistent, since all points were measured with the same instrumental resolution. However, it is important to realize that lacking a quantitatively measure of the effect of this resolution, i.e. lacking detailed knowledge of the line shape, the vertical scale in Fig. 2 has no quantitative meaning. The error bars shown in Fig. 2 require discussion. They were assigned by making repeatability measurements as well as by testing the results of repositioning the cursors presumably located at the maximum of the peak and near the base of the peak at a place which we hoped would represent the "average" noise power density. The largest amount of scatter by far was incurred with the repeatability measurements. The reason is that the modulation peaks are only a few points wide, so that small variabilities from sample-to-sample in, for example the signal generator (modulation) frequency, resulted in large variations in the peak amplitudes as the modulation power density happened to be shared among a few or several "bins" as the Fourier transform was processed. We suggest that the comparably large error bars in the ring laser experiment[7] might have derived from the same variability.

In their original paper, McNamara, *et al*[7] outlined a theory which has now been written up in detail[12]. We quote here only the result:

$$S/N = (c/D^2)\exp(-2\Delta U/D), \tag{4}$$

where S/N is the power density amplitude ratio and c is some constant. Following the usual definition, the SNR in decibels (db) is given by SNR = 10log(S/N). As did the authors of Ref. 7, we have found it necessary to add an offset value $D_0 = 0.032$ v^2/Hz to all values of D applied to the simulator in order to fit the data with Eq. (4). Presumably, this represents the effect of the internal circuit noise. For c = 60.8 and $\Delta U = 0.5$ the theory is represented by the curve shown in Fig. 2 which is qualitatively comparable to that obtained in the Schmidtt trigger and ring laser experiments.

We conclude by emphasizing that ideally the power spectra of stochastic resonance systems with additive modulation contain delta functions which are rendered into measurable peaks by experimental systems with finite frequency resolution. Quantitative measurements of the SNR of such systems can be obtained by introducing the details of the instrumentally broadened line shape, or alternatively by integrating the measured power spectra thus transforming the peaks into steps whose amplitudes are independent of the line shapes.

We are grateful to Kurt Wiesenfeld for a valuable discussion. Thanks are also due to Peter Hanggi for remarks concerning the Four'er transforms of modulated systems. This work was supported by the Office of Naval Research grant no. N00014-88-K-0084.

REFERENCES

*present address: Department of Electrical Engineering; Michigan State University; East Lansing, MI. 48823

+permanent address: Institute of Semiconductors; Chinese Academy of Sciences; Beijing, People's Republic of China.

1. R. Benzi, A. Sutera and A. Vulpiani, J. Phys. A. **14**, L453 (1981)

2. C. Nicolis, Tellus **34**, 1 (1982)

3. R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, Tellus, **34**, 11 (1982)

4. C. Nicolis and G. Nicolis, Tellus **33**, 225 (1981)

5. A. Sutera, Quart. J. Roy. Meteorl. Soc. **107**, 137 (1981)

6. S. Fauve and F. Heslot, Phys. Lett. **97A**, 5 (1983)

7. B. McNamara, K. Wiesenfeld and R. Roy, Phys. Rev. Lett. **60** 2626 (1988)

8. The usual engineering definition of the SNR is the ratio of the total *energy* of the signal to that of the noise. For band limited signals and noise, the total energies are obtained by integrating the power spectra so that the delta functions become finite amplitude step discontinuities.

9. There are two additional limiting cases: The first is multiplicative noise with either multiplicative or additive modulation, wherein the barrier height is made noisy. In this case, however, there are no noise induced switching events, since the state at $x = 0$ is stabilized by the noise; see M. Lücke and F. Schank, Phys. Rev. Lett. 54, 1465 (1985). The second is multiplicative noise on the amplitude of the modulation itself, which can be disregarded since, in this case, the power spectrum of the modulation is broadened *before* being introduced into the system. Of course, in a more complicated Langevin equation, the noise may multiply one or more nonlinear terms (in addition to the term which determines the barrier height) but can always be transformed to an equivalent additive noise.

10. J.-P. Eckmann and L. E. Thomas, J. Phys. A, 15, L261 (1982)

11. R. Fox, preprint.

12. B. McNamara and K. Wiesenfeld, preprint, submitted to Phys. Rev. A.

13. R. Roy, P. A. Schulz and A. Walther, Opt. Lett. 12, 672 (1987)

14. J. Smythe, F. Moss and P. V. E. McClintock, Phys. Rev. Lett. 51, 1064 (1983)

15. The dimensionless noise correlation time (the ratio of the noise correlation time to the integrator time constant, or characteristic time, of the simulator) in this experiment was 0.20, which is not quite small enough to well approximate white noise.

FIGURE CAPTIONS

Fig. 1. Example results of the analog simulation. (a) A time series for Eq. (1) with a = $(2)^{1/2}$, D = 0.20 v^2/Hz, and τ = 0.20. (b) The 500 Hz modulation which also establishes the time scale for (a) with ϵ = 0.4. (c) An example power spectrum which is the result of 200 averages computed from individual time series obtained for the conditions stated in (a) and (b). The sharp peak at 500 Hz establishes the frequency scale. (d) A power spectrum obtained for the same conditions as listed in (a) except from the simulator of Eq. (2), and only 100 averages were accumulated.

Fig. 2. The measured SNR versus $D^{1/2}$ from the simulator of Eq. (1) with a = $(2)^{1/2}$ (which implies that ΔU = 0.5), τ = 0.20 and the modulation as in Fig. 1 (b). For a discussion of the error bars, see the text. The curve is Eq. (4) with c = 60.8 and ΔU = 0.5.

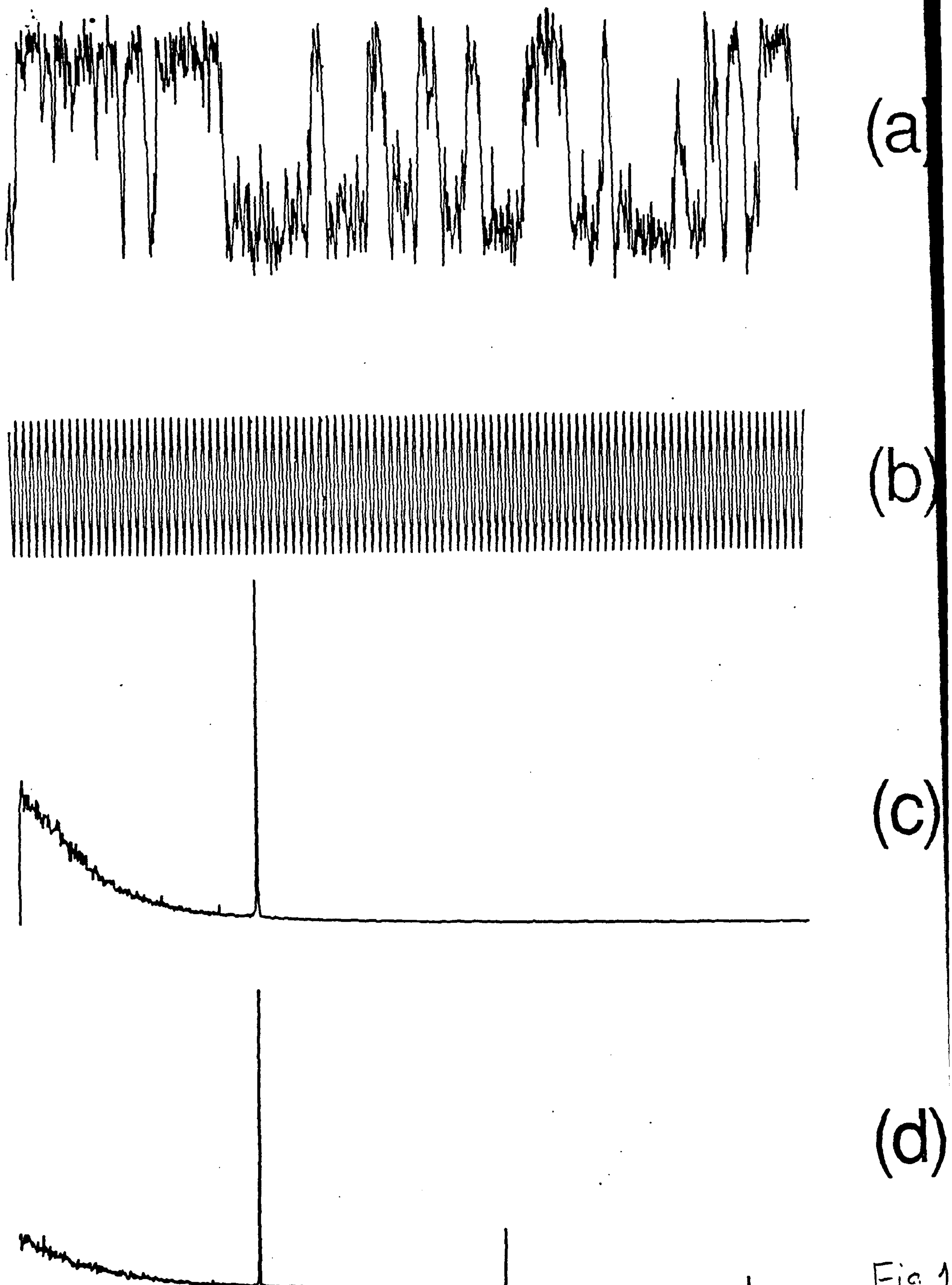


Fig 1

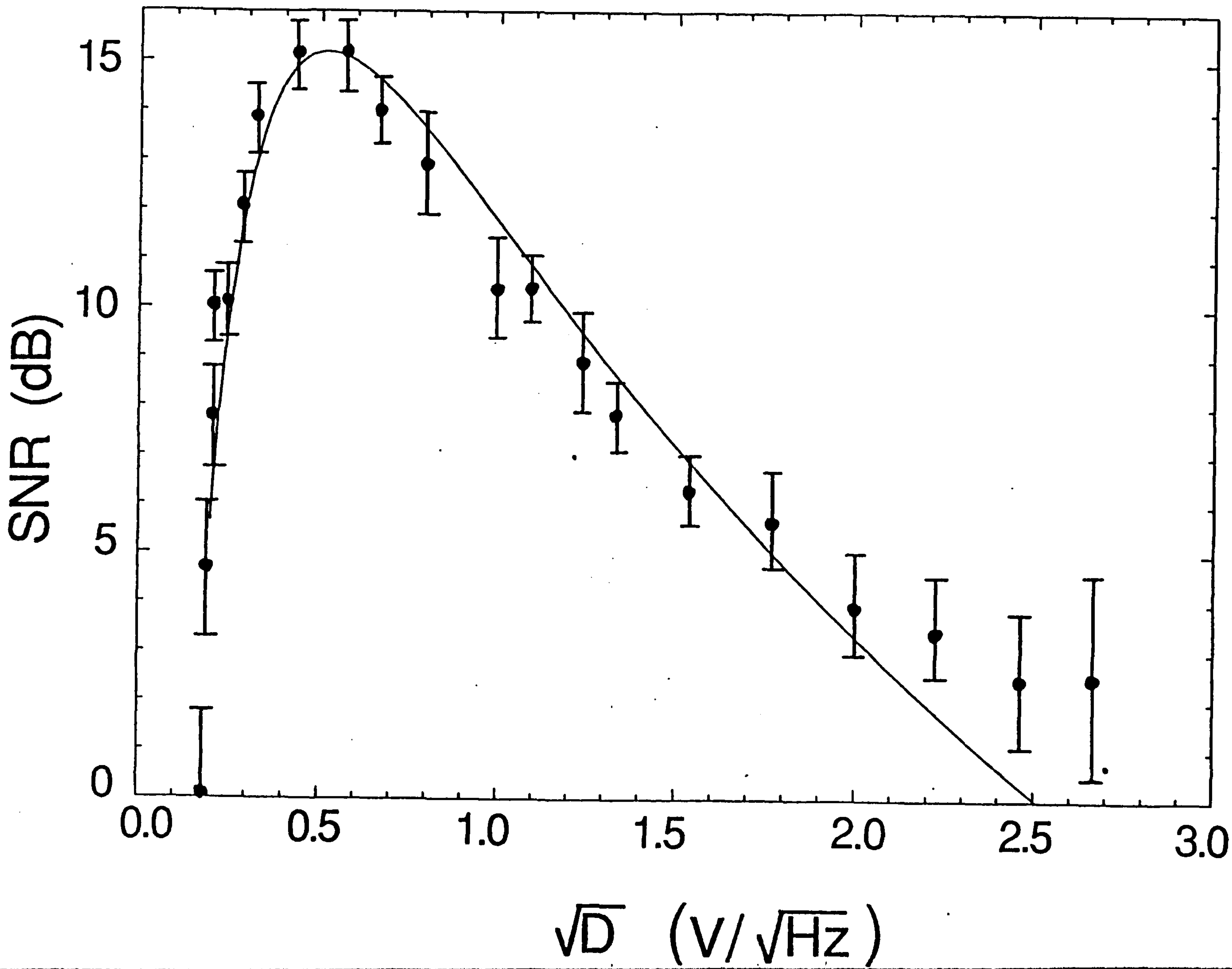


Fig. 7

Submitted J. Stat. Phys

Analog simulation of a simple system with state dependent diffusion

K. Sinha and Frank Moss
Department of Physics
University of Missouri at St. Louis
St. Louis, MO 63121

ABSTRACT

We have constructed an electronic simulator of a simple bistable system driven by noise, whose intensity is determined by the instantaneous value of the coordinate. We observe that the most probable state of the system can be reversed by altering the noise intensity only in the neighborhood of the barrier: an effect pointed out by Landauer many years ago in the context of discussions on entropy related stability criteria for nonequilibrium systems. We compare detailed measurements on the system with the recent white noise calculations of Landauer and van Kampen. The system also has interesting possibilities for tests of contemporary colored noise theory which we illustrate with an example.

PACS 05.40.+j

I. *Introduction*

Many years ago Landauer raised a discussion on the implications of state dependent diffusion in the context of entropy generation and stability in nonequilibrium systems.[1] His point was that the relative stability of a multistable system could be altered by path dependent diffusion even when localized to the neighborhoods of the potential barriers separating the deterministically stable or metastable modes. That is, the most probable state of such a system cannot be determined from information about the local minima alone. Later it was observed that in certain models, state dependent diffusion can even generate maxima in the probability distribution at locations where no potential minima exist.[2] These early studies gave rise to wide discussions which continue to be of current interest.[3]

The more general problem of diffusion in inhomogeneous media has recently been carefully analyzed by van Kampen.[4] State dependent diffusion which is periodic and commensurate with a spatially periodic potential can give rise to continuous currents in the absence of externally applied fields. Such systems have recently been treated by Buttiker[5] and van Kampen[6] as well as Landauer.[7]

In this paper we consider the simplest model for state dependent diffision in a bistable system as originally suggested.[1] We take the standard quartic as the potential

$$V(x) = -x^2/2 + x^4/4 + \varepsilon x, \tag{1a}$$

with the dynamics being determined by the infinitely damped system

$$\dot{x} = x - x^3 + g(x)\xi(t) - \varepsilon, \tag{1b}$$

where $\xi(t)$ is a Gaussian, white noise with zero mean and correlation $\langle\xi(t)\xi(s)\rangle = 2D_n\delta(t-s)$; and where $g(x)$ is defined by

$$\begin{aligned} g(x) &= 1, && \text{for } x < 0 \text{ or } x > 1/2 \\ g(x) &= 1 + \Delta, && \text{for } 0 < x < 1/2. \end{aligned} \tag{2}$$

The noise intensity is D_n and ε controls the symmetry of the potential wells. This specific system has recently been analyzed by both van Kampen,[6] who, in a notably clear tutorial, derived the probability densities from thermodynamic arguments as well as from the diffusion equation, and by Landauer.[7]

The effect of the reversal of stability in this bistable system due to an increased temperature (larger D) in the neighborhood of the barrier is illustrated in Fig. 1. Though the illustration is schematic, the graphs are quantitative. On the left is shown U(x) as given by Eq. (1a) with $\varepsilon = 0.05$. On the right are shown the probability densities as measured on the simulator described in Section III below for increasing values of Δ. The effect of Δ in reversing the most probable state is clearly evident.

This paper is organized as follows: In Section II the theory is briefly reviewed. Expressions for the ratio of the amplitudes of the stationary probability densities in the two wells and for the magnitudes of the discontinuities at the locations where the noise intensity is discontinuously changed are summarized. In Section III the simulator and the methods of measurement are described. The main results are presented in Section IV where measurements of the amplitude ratio of the probability density and the magnitudes of the discontinuities are compared to the theory of Refs. 6 and 7. In Section V we examine the colored noise problem and the possibility that this system could be of use in the testing of contemporary colored noise theory. As examples, data on the amplitude ratios are compared to the predictions of the conventional small correlation time theory, as recently improved by Fox,[8] and to Hanggi's *ansatz*.[9] The advantage of this system is that the results depend only on the *ratios* of probability densities, so that the correlation time dependent prefactors which often appear in various forms in colored noise approximate theories *cancel out*, revealing the exponential behavior alone.

This could be a considerable advantage, since it is often difficult or impossible to clearly distinguish between the influence of the prefactors and exponentials using numerical, matrix continued fraction or analog simulations. A vigorous discussion on the merits and accuracy of various colored noise approximate theories is currently in progress.[10,11] Finally in Section VI we summarize our results.

II. Theory

We consider the symmetric potential defined by Eq. (1a) with $\varepsilon = 0$ and as shown in Fig (2b). Following Landauer's notation,[7] the left hand well is located at A ($x = -1$), the barrier at B ($x = 0$), the region of increased noise intensity between B and C ($x = 1/2$) and the right hand well at D ($x = 1$).

It is necessary to first consider what happens at a temperature (or noise intensity) discontinuity. For stationary conditions the probability currents across the discontinuity at B, for example, can be written $\rho(B^-)v(B^-) = \rho(B^+)v(B^+)$, where ρ is the probability density and v is the velocity of a particle in thermal equilibrium at temperature T just to the left of the discontinuity (B^-) or just to the right (B^+). The velocity is $v \propto \sqrt{T}$, so that $\rho(B^+)/\rho(B^-) = (T_L/T_H)^{1/2}$, where T_L is the (lower) temperature outside the region BC and T_H is the (higher) temperature inside BC. We identify the temperature with the diffusion or noise intensity through $D_n = \mu kT$, and since we consider only homogeneous media we take μ = const. = 1. The ratios are thus

$$\rho(B^+)/\rho(B^-) = \rho(C^-)/\rho(C^+) = (D_{nL}/D_{nH})^{1/2}. \qquad (3)$$

Landauer then argues that the densities within the regions are given by the Boltzmann distribution $\exp(-U(x)/D_n)$. In addition to Eq. (3), the ratios are $\rho(B^-)/\rho(A) = \exp[-(U_B-U_A)/D_{nL}]$; $\rho(C_-)/\rho(B^+) = \exp[-(U_C-U_B)/D_{nH}]$; and $\rho(D)/\rho(C^+) = \exp[-(U_D-U_c)/D_{nL}]$. These probabilities are then multiplied together, whereupon the ratios given by Eq. (3) cancel, and the result is

$$\rho(D)/\rho(A) = \exp[-(U_D-U_A)/D_{nL}] \exp[-\Delta U(D_{nL}-D_{nH})/D_{nL}D_{nH}], \tag{4}$$

where $\Delta U = U_C - U_B$. van Kampen[6] obtains the same result for the amplitude ratio at the wells, but he predicts that for ratio at the discontinuities

$$\rho(B^+)/\rho(B^-) = \rho(C^-)/\rho(C^+) = D_{nL}/D_{nH} \tag{5}$$

instead of Eq. (3). In this work, we consider only the symmetric ($\varepsilon = 0$) potential. With $U_A = U_D$, Eq. (4) becomes

$$\rho(D)/\rho(A) = \exp[-\Delta U(D_{nL}-D_{nH})/D_{nL}D_{nH}]. \tag{6}$$

III. The Simulator

Figure 2(a) shows a schematic diagram of the simulator of Eqs. (1) and (2). The design is straight-forward and one which we have used before [9,12] with the exception of the added system for generating g(x). This is shown by the two comparitors which continuously test the voltage on x and compare it to the preset values x_0 and x_1 which mark the boundries.of the region of increased noise intensity. Each time the trajectory crosses one of the boundries an output is applied to the AND gate shown. This gate, in turn, and the adder which follows it, provide a suitable voltage V_g, to a multiplier whose other input is the noise voltage V_n. The logic is arranged in such a way that $V_g = 1 + \Delta$ when $x_0 < x(t) < x_1$, and $V_g = 1$ when x(t) is outside this range, or $V_g \equiv g(x)$. The noise voltage $\xi(t)$ which drives the trajectory in the bistable potential is then defined by

$$\xi(t,x) = g(x)V_n(t) \ , \tag{7}$$

with g(x) defined by Eqs. (2).

The noise voltage is supplied by a noise generator[13] of wide but finite bandwidth, and is therefore necessarily colored. In order to define its correlation time τ_n precisely, it is passed through a linear, single pole filter with transfer function $H(\omega) = 1/[1 + (\omega\tau_n)^2]$. The simulator scales

time with the integrator time constant τ_i, so that the dimensionless correlation time is $\tau = \tau_n/\tau_i$, and the correlation function of V_n is

$$<V_n(t)V_n(s)> = (D_n/\tau)\exp(-|t - s|/\tau) \tag{8}$$

In the limit $t \to s$, the noise intensity is defined by

$$D_n = \tau < V_n^2 >, \tag{9}$$

and the mean square noise voltage is a measured quantity. In this simulation τ_i = 100 μs and the range of τ_n was 20 μs to 500 μs (the range of τ was then 0.20 to 5). For comparison with the white noise theories, however, we set the value τ = 0.20. Equations (2), (7) and (9) then give for the noise intensities

$$D_{nL} = \tau < V_n^2 >$$

$$D_{nH} = \tau < V_n^2 > (1 + \Delta)^2, \tag{10}$$

Figure 2(b) shows the potential of Eq. (1a) with $\varepsilon = 0$. The region BC of enhanced noise intensity is shown by the darkened segment which lies between $x_0 = 0$ and $x_1 = 1/2$. The potentials shown in Eq. (4) are then given by $U_A = -1/4$, $U_B = 0$, $U_C = 7/64$ and $U_D = 1/4$, as determined by Eq. (1a) with $\varepsilon = 0$. The theoretical prediction for the amplitude ratio of the densities in the wells, Eq. (4), then becomes

$$\rho(D)/\rho(A) = \exp\left\{- \frac{7}{64D_{nL}}\left[1 - \frac{1}{(1+\Delta)^2}\right]\right\}. \tag{11}$$

Using Eqs. (3), and (10) the predictions for the amplitude ratios at the discontinuities are

$$\rho(B^+)/\rho(B^-) = \rho(C^-)/\rho(C^+) = \begin{cases} 1/(1+\Delta) & \text{(12a)} \\ \text{or} & \\ 1/(1+\Delta)^2 & \text{(12b)} \end{cases}.$$

The simulator is operated by first setting $\Delta = 0$ and by adjusting ε in order to obtain a symmetric density ρ. Checks on this symmetry adjustment were frequently made throughout the experiment. Figure 3 shows an example

set of measured densities for increasing values of Δ commencing with the symmetric density shown at the top. These densities were assembled as averages from a sequence of time series x(t) of 4096 digitized points each. Typically 800 such time series were obtained so that each density resulted from about 3.2×10^6 digitized points. For each density, the amplitudes at the two peaks and at the high and low sides of each discontinuity were recorded.

IV. The results

Figure 4 shows the results of our measurements of the amplitude ratio in the wells as a function of Δ for various values of D_n. The solid lines are plots of Eq. (11) with D_{nL} determined directly from τ and measurements of $< V_n^2 >$. There are no adjustable constants. The agreement between Eq. (11) and our measurements is excellent, but probably fortuitous, since the noise driving our simulator is actually colored with $\tau = 0.2$, and the theory is valid only for white noise. The region of quasi-white behavior of the simulator is $\tau << 1$, but in practice the usable dynamic range of the analog components imposes a lower limit of $\tau \sim 0.1$ to 0.2 for reliable operation without voltage clipping. Nevertheless the systematic behavior with D_{nL} and Δ is convincing.

The scatter in the data points shown on Fig. 4 is the result of both systematic and statistical errors. The ratio $\rho(D)/\rho(A)$ is obtained from two large numbers, each with a certain statistical error. The statistical scatter was estimated from short term repeatability measurements, and is shown as the example error bar on the $D_{nL} = 0.36$ data. A more troublesome error stems from longer term variations in $< V_n^2 >$ due to very low frequency drifts, or systematic variations, in the noise generator. An attempt has been made to estimate the maximal effects of this error, and is shown by the dashed curves for $D_{nL} = 0.40$ (upper curve) and $D_{nL} = 0.32$ (lower curve).

We have also measured the amplitudes of the density on both sides of the discontinuities in order to find the ratios $\rho(B^+)/\rho(B^-)$ and $\rho(C^-)/\rho(C^+)$. Both Refs. (6) and (7) make specific predictions for the behavior of these ratios with Δ as shown by Eqs. (12). In Ref. (7) it is argued additionally that whatever the functional dependence of the magnitude of the discontinuities on Δ, it can be expected to be the same for both discontinuities, and therefore will cancel in the result Eq. (11). Our measurements of these ratios are shown on Fig. 5, where (a) and (b) display the results for a larger and a smaller noise intensity respectively. The statistical errors on these data are relatively larger, as can be judged from the scatter, because the discontinuities occur at low amplitude on the density. The theoretical predictions are shown by the curves: Ref. (7) and Eq. (12a) are the solid curves, and Ref. (6) and Eq. (12b) are the dashed curves.

We remark that both theories predict the equality of the magnitudes of the two discontinuities, and their independence on D. Certainly to within the scatter of our data the measured magnitudes appear to be equal as shown by the solid circles (discontinuity at B, $x = 0$) and the open circles (discontinuity at C, $x = 1/2$). Both sets of data seem to lie systematically higher than Eq. (12a), a trend which seems more evident for large noise intensity. Nevertheless, the systematics clearly favor Eq. (12a).

Certainly these results come as no surprise to anyone, since exact solutions of white noise systems in the limit of large damping are very well known. The quasi-white noise measurements shown here serve to indicate the accuracy of our simulator.

V. Colored Noise

Nearly all colored noise approximate theories begin with an effective Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial x}\left[U'(x)\rho(x,t)\right] + \frac{\partial^2}{\partial x^2}D(x)\rho(x,t), \tag{13}$$

wherein the diffusion $D(x) \rightarrow D(x,\tau)$, is intended to approximately account for the nonzero point-to-point correlations over x which are induced by a correlated driving force (the colored noise). [The case wherein $D(x)$ is the result of an inhomogeneous medium has been recently examined by van Kampen[4]]. The recent rapid growth of the theoretical colored noise industry[8-11] has resulted in a variety of expressions for the "renormalized" diffusion $D(x,\tau)$.

For the purpose of example we choose here only two: the improved "small τ" approximation of Fox[8,14] which reduces to the often cited result of Sancho, et al[15] in the limit of small τ; and the ansatz due to Hanggi.[9] The Fox results are obtained from Eq. (13) with

$$D_F(x,\tau) = D_0[1 + \tau\, U''(x)]^{-1}\quad , \tag{14}$$

which results in the stationary density

$$\rho_F(x) = (1 - \tau + 3x^2\tau)\exp[-\, U_F(x,\tau)/D_0]\ , \tag{15a}$$

with $U_F(x,\tau)$ for our potential given by

$$U_F(x,\tau) = -\, x^2/2 + x^4/4 + \tau\,(x^2/2 - x^4 + x^6/2) \tag{15b}$$

Hanggi's results are obtained from Eq. (13) with

$$D_H(x,\tau) = D_0[1 + \tau(3\langle x^2\rangle - 1)]^{-1}\quad , \tag{16}$$

where $\langle x^2 \rangle$ is an average over the trajectories, and this $D(x)$ is specific to our potential. The density in this case is

$$\rho_H(x) = N\exp[-U(x)/D_H(x,\tau)] \tag{17}$$

where $U(x)$ is given by Eq. (1a).

As a zeroth order approximation, we can simply substitute these results into the white noise formula for the ratio of the densities. This procedure is based on the observation that, even though this is a system with state dependent noise, the space is separated into three regions, within each of which the noise intensity is constant. Nevertheless, to simply substitute colored noise densities for the white ones neglects the effects (if any) of correlations across the boundries. Our only justification for this procedure is that our measurements of the amplitude ratios of the discontinuities, as discussed below, show no systematic behavior with τ to within our (not so small) statistical errors.

Using Eqs. (15)-(17), results in the following predictions

$$\rho(D)/\rho(A)_F = \exp\{(\frac{1}{D_{nL}})[1 - \frac{1}{(1+\Delta)^2}][- \frac{7}{64} + \frac{9\tau}{128}]\} \tag{18}$$

for the Fox theory; and

$$\rho(D)/\rho(A)_H = \exp\{- \frac{7(1+2\tau)}{64D_{nL}}[1 - \frac{1}{(1+\Delta)^2}]\} \tag{19}$$

for the Hanggi theory. We note that the τ dependent prefactors (if any) on the densities cancel out in this application. Further, these two expressions make qualitatively opposite predictions: Eq. (18) shows the ratio increasing with τ, while Eq. (19) shows it decreasing. Both show the same Δ dependence (which is also the white noise Δ dependence).

Our measurements at fixed $\Delta = 1$ are shown in Fig. 6 where the open circles are for the large D_{nL} and the closed circles are for the small D_{nL}. Equation (18) is shown by the dashed curve and Eq. (19) by the solid curve. Both are plotted for $D_{nL} = 0.40$ which is also the noise intensity for which the solid circles were obtained. The lozenge on the vertical axis shows the white noise limit for all theories and illustrates the difficulty in

comparing our simulations with white noise results: the quasi-white noise simulation was done for $\tau = 0.2$ but does not extrapolate well to the lozenge shown on the axis at $\tau = 0$.

Finally, measurements of the amplitude ratios at the discontinuities versus τ for $\Delta = 1$ are shown in Fig. 7. There is no discernible τ dependence of these ratios.

VI. Summary and conclusions

We have measured stationary density amplitude ratios for a system with state dependent noise applied in d of three discrete regions. The noise intensity was changed discontinuously at the boundries of the inner region. The measured results are in good agreement with the predictions of white noise calculations due to Landauer and van Kampen. In addition, we have repeated two sets of amplitude ratio measurements for a range of noise correlation times, and compared the results to two current colored noise approximate theories. While neither approximation accurately describes the data, the *ansatz* of Ref. 9 is in better qualitative agreement.

We are grateful to R. Landauer and M. Buttiker for stimulating discussions and for suggesting this experiment. We are also indebted to R. Fox, P. Hanggi and J. M. Sancho for various discussions. This work was supported by the Office of Naval Research, Grant No. N00014-88-K-0084.

REFERENCES

1. R. Landauer, Phys. Rev. A 12, 636 (1975); and in the Maximum Entropy Formalism edited by R. D. Levine and M. Tribus (MIT Press, Cambridge, MA 1978) p. 321.
2. W. Horsthemke and R. Lefever Phys. Lett. 64A 19 (1977) and Noise Induced Transitions (Springer Verlag, Berlin 1984).
3. See, for example, B. J. West and K. Lindenberg, in Studies in Statistical Mechanics Vol. XIII, edited by J. L. Lebowitz (North-Holland, Amsterdam 1987) chapt. 2.
4. N. G. van Kampen, Z. Phys. B 68 135 (1987).
5. M. Buttiker, Z. Phys. B 68 161 (1987).
6. N. G. van Kampen, IBM J. Res. & Dev. 32 xxx (1988).
7. R. Landauer, J. Stat. Phys., to be published.
8. R. Fox, Phys. Rev. A 37, 911 (1988)
9. P. Hanggi, T. J. Mrockowski, F. Moss and P.V.E. McClintock Phys. Rev. A32 695 (1985)
10. for a collection of reviews, see, Noise in Nonlinear Dynamical Systems, Vol. I, edited by F. Moss and P.V.E. McClintock (Cambridge University Press, Cambridge, forthcoming, 1988).
11. in addition to Refs. 8-10, see also:

    P. Jung and P. Hanggi Phys. Rev. A35, 4464 (1987) and preprint.

    F. Marchesoni, Phys. Rev. A36 4050 (1987); Th. Leiber, F. Marchesoni and H. Risken, Phys. Rev. Lett. 59, 1381 (1987)

    J. Masoliver, K. Lindenberg and B. J. West, Phys. Rev. A35 3086 (1987)

    C. R. Doering, P. S. Hagan and C. D. Levermore, Phys. Rev. Lett. 59 2129 (1987)

M. M. Dygas, B. J. Matkowsky and Z. Schuss, SIAM J. Appl. Math., in press and to be published.

S. Faetti and P. Grigolini, Phys. Rev. A36 441 (1987)

L. Schimansky - Geier, Phys. Lett. A 126 455 (1988)

V. Altares and G. Nicolis, J. Stat. Phys. 46 191 (1987) and preprint.

J. M. Sancho, preprint.

12. F. Moss and P.V.E. McClintock, Z. Phys. B 61 381 (1985).

13. Quan Tech model 420.

14. R. F. Fox, Phys. Rev. 33A 467 (1986) and Phys. Rev. 34A 4525 (1986)

15. J. M. Sancho, M. San Miguel, S. L. Katz and J. D. Gunton, Phys. Rev. A26 1589 (1982).

FIGURE CAPTIONS

Fig. 1 Potentials from Eq. (1) with $\varepsilon = 0.05$ are shown on the left and measured densities on the right for $\Delta = 0$, 0.45, and 1.3 top-to-bottom, showing the reversal of the most probable state.

Fig. 2 (a) A schematic diagram of the simulator, showing multipliers (x) summers (±), comparitors (comp.) and an and circuit (AND). (b) An example of the potential utilized in the simulator with the region of increased noise intensity BC shown in bold lines between $x_0 = 0$ and $x_1 = 1/2$.

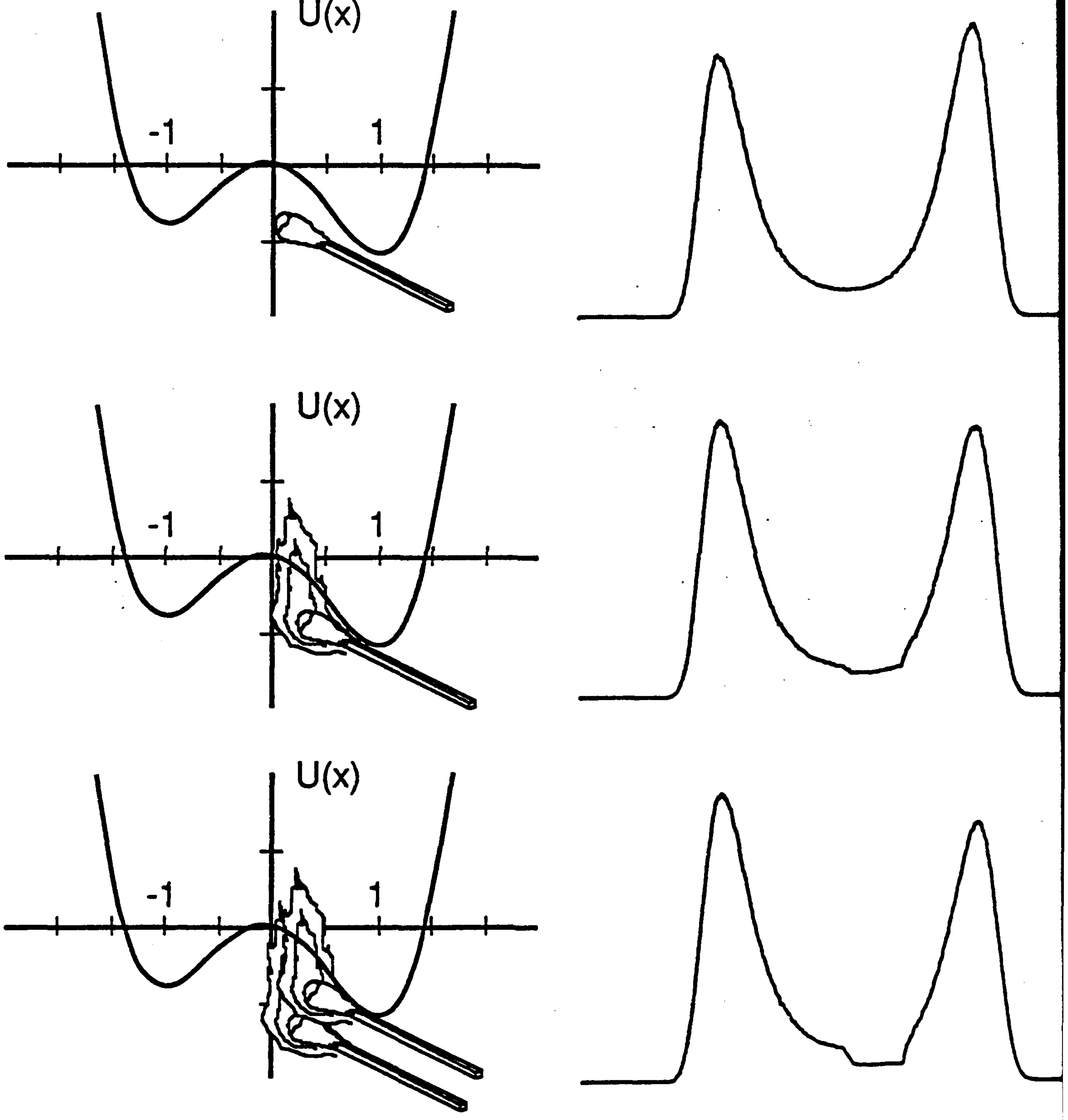
Fig. 3 Probability densities measured for values of $\Delta = 0$, 0.5, 0.75, and 1.0 top-to-bottom. The region of increased noise intensity BC is clearly evident for $\Delta > 0$.

Fig. 4 $\rho(D)/\rho(A)$ versus Δ for values of $D_{nL} = 0.20$ (squares), 0.36 (triangles), 0.60 (circles) and 1.6 (inverted triangles). The solid lines are plots of Eq. (11). The dashed lines estimate the systematic error in D_{nL}.

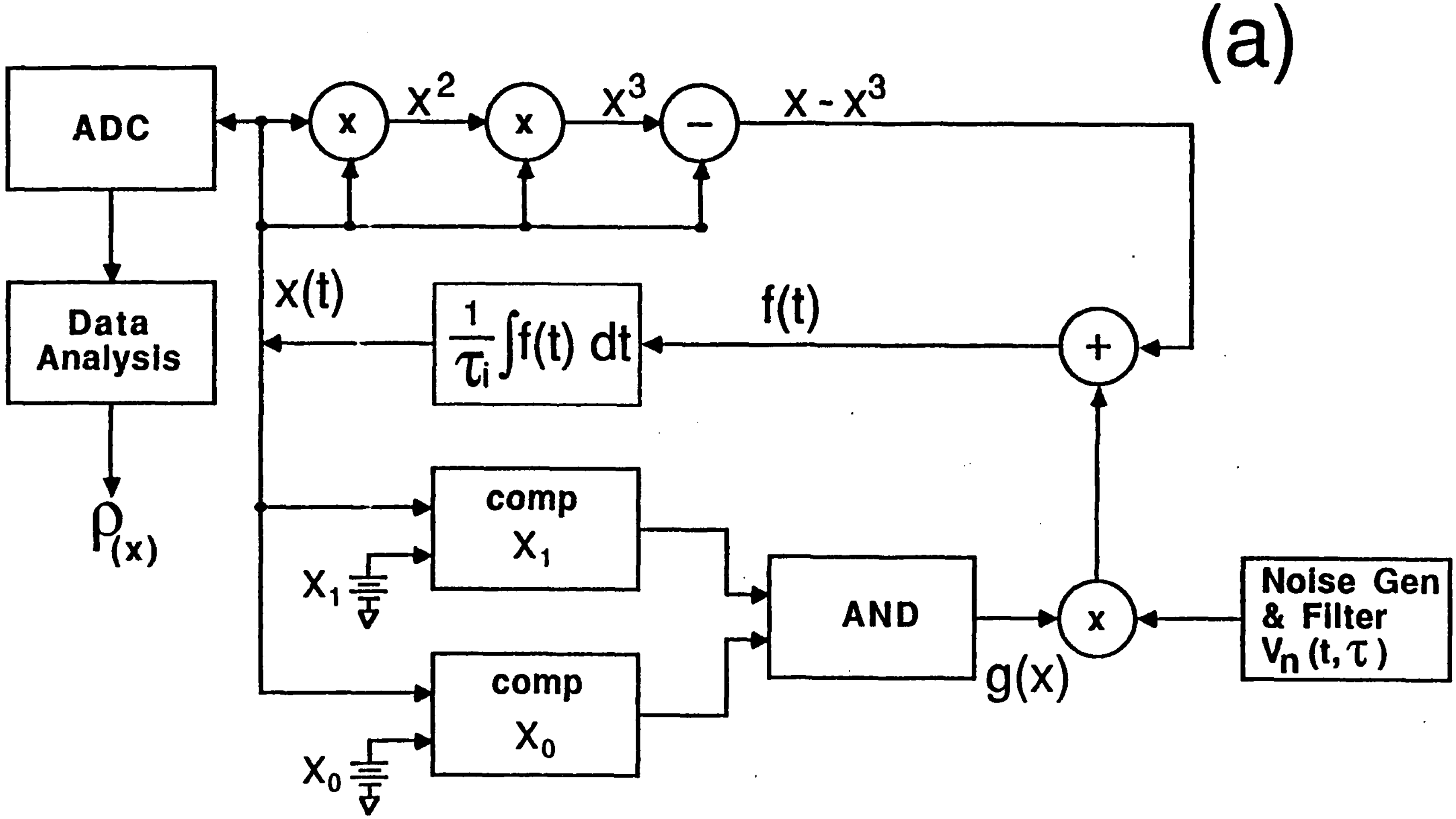
Fig. 5 $\rho(B^+)/\rho(B^-)$ (solid circles) and $\rho(C^-)/\rho(C+)$ (open circles) for (a) $D_{nL} = 1.60$ and (b) $D_{nL} = 0.36$. The solid line is Eq. (12a) and the dashed line is Eq. (12b).

Fig. 6. The effect of colored noise is shown by this plot of $\rho(D)/\rho(A)$ versus τ for $\Delta = 1$. The open circles are for $D_{nL} = 1.6$ and the solid circles are for $D_{nL} = 0.40$. The dashed curve is Eq. (18) and the solid curve is Eq. (19), both for $D_{nL} = 0.40$.

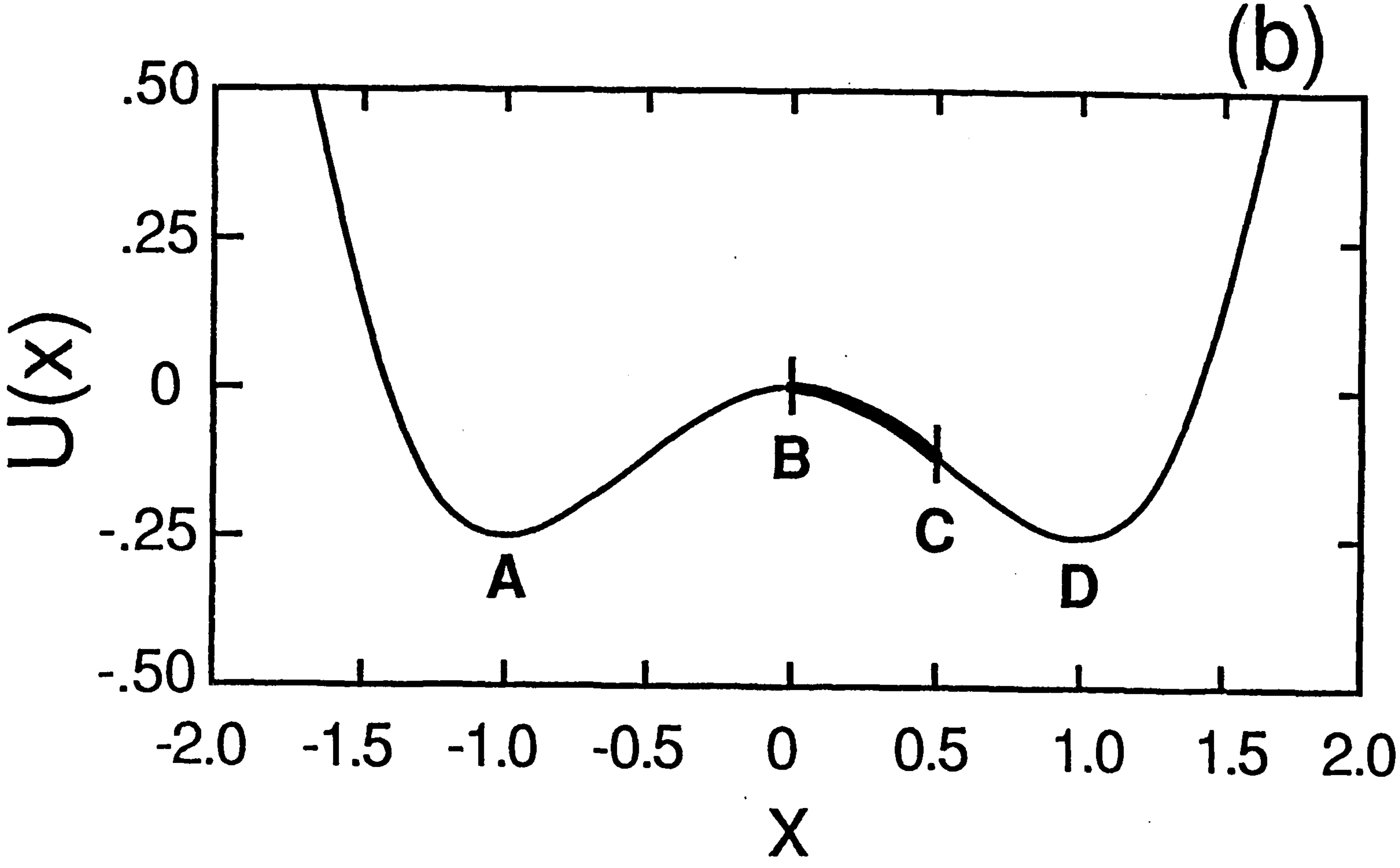
Fig. 7. The effect of noise color τ on the discontinuities $\rho(B^+)/\rho(B^-)$ (solid circles) and $\rho(C^-)/\rho(C^+)$ (open circles). The upper data set is for $D_{nL} = 1.60$ and the lower set for $D_{nL} = 0.40$. All data are for $\Delta = 1$.

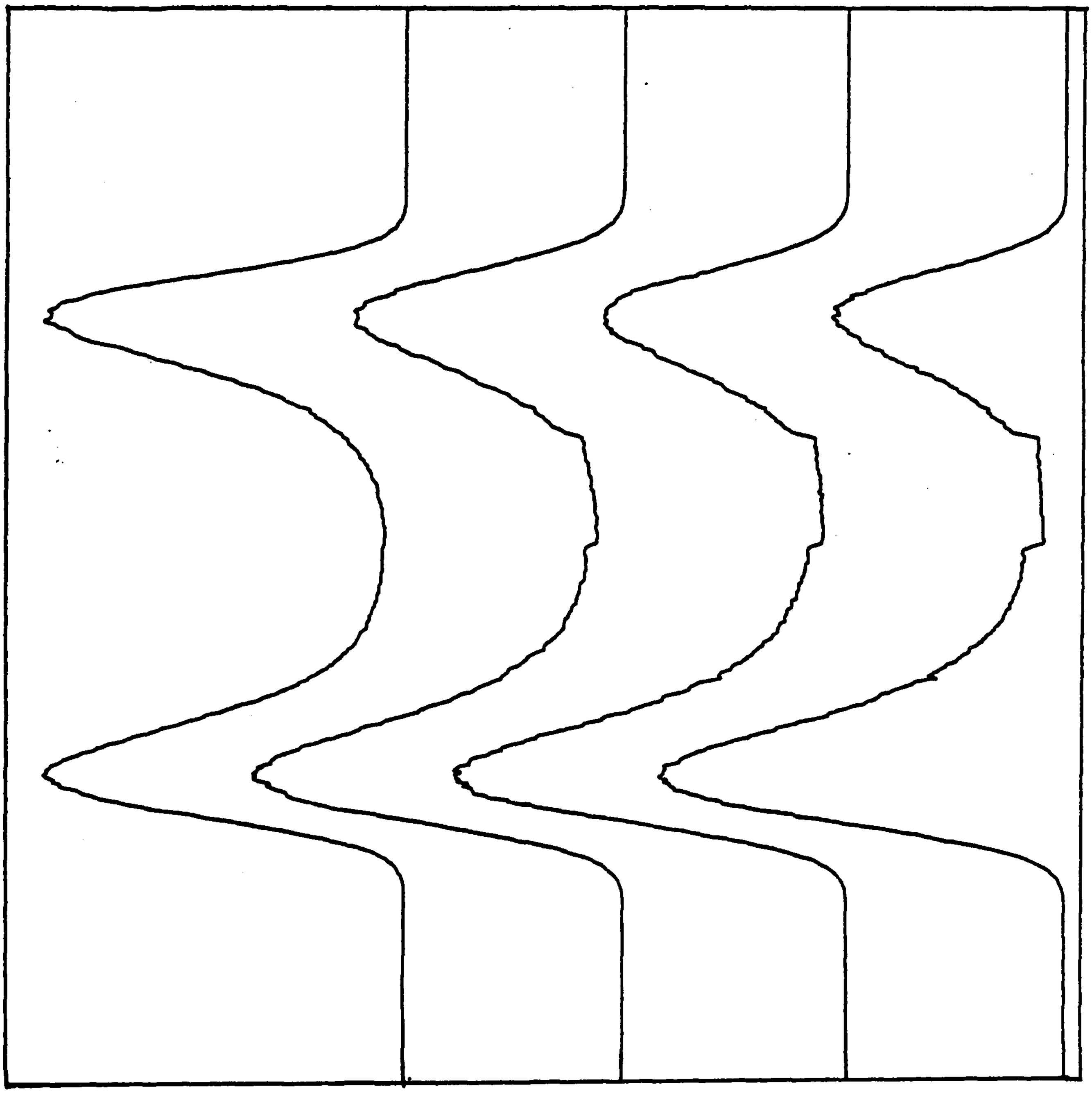


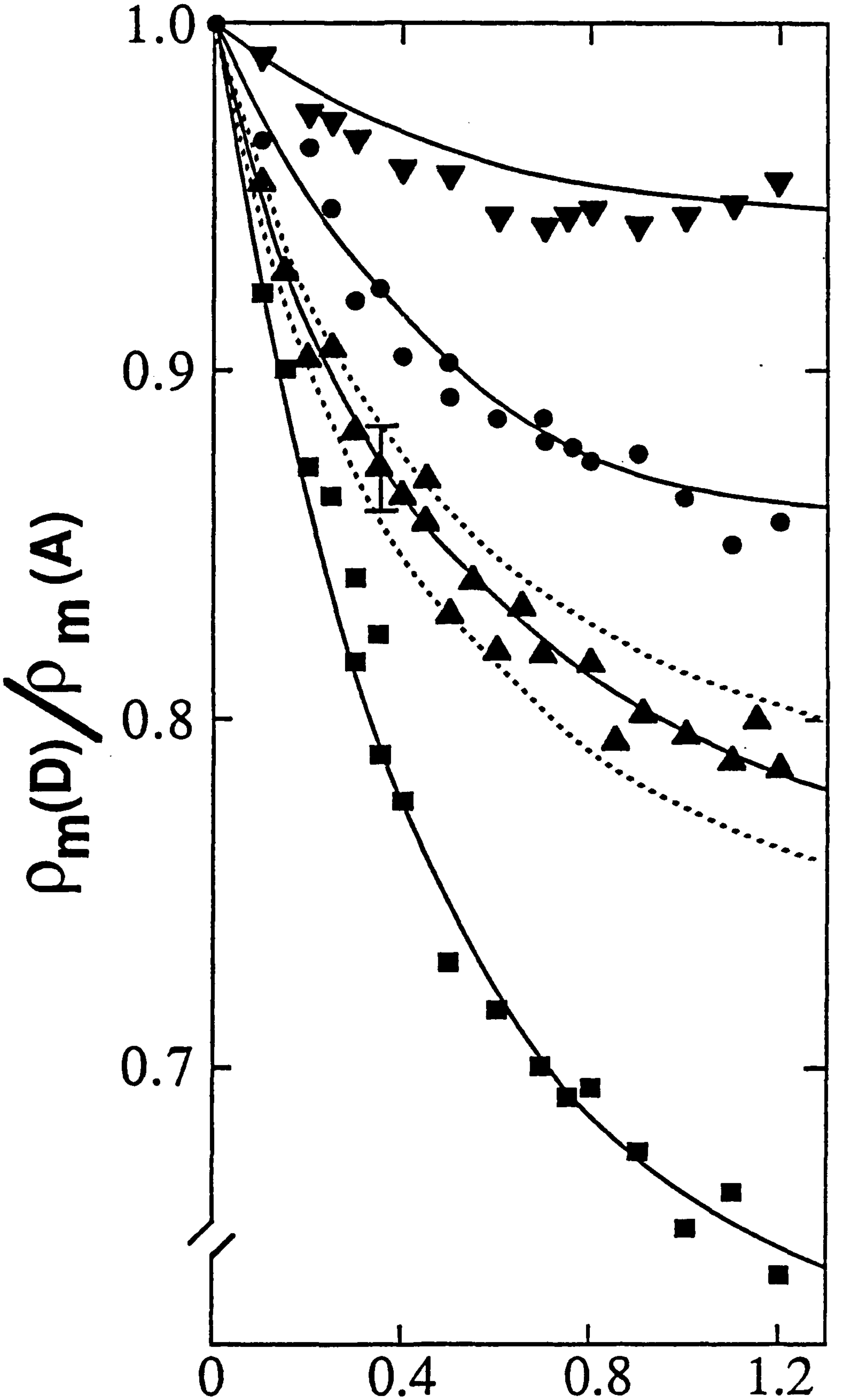
U(x)
-1
1
U(x)
-1
1
U(x)
-1
1



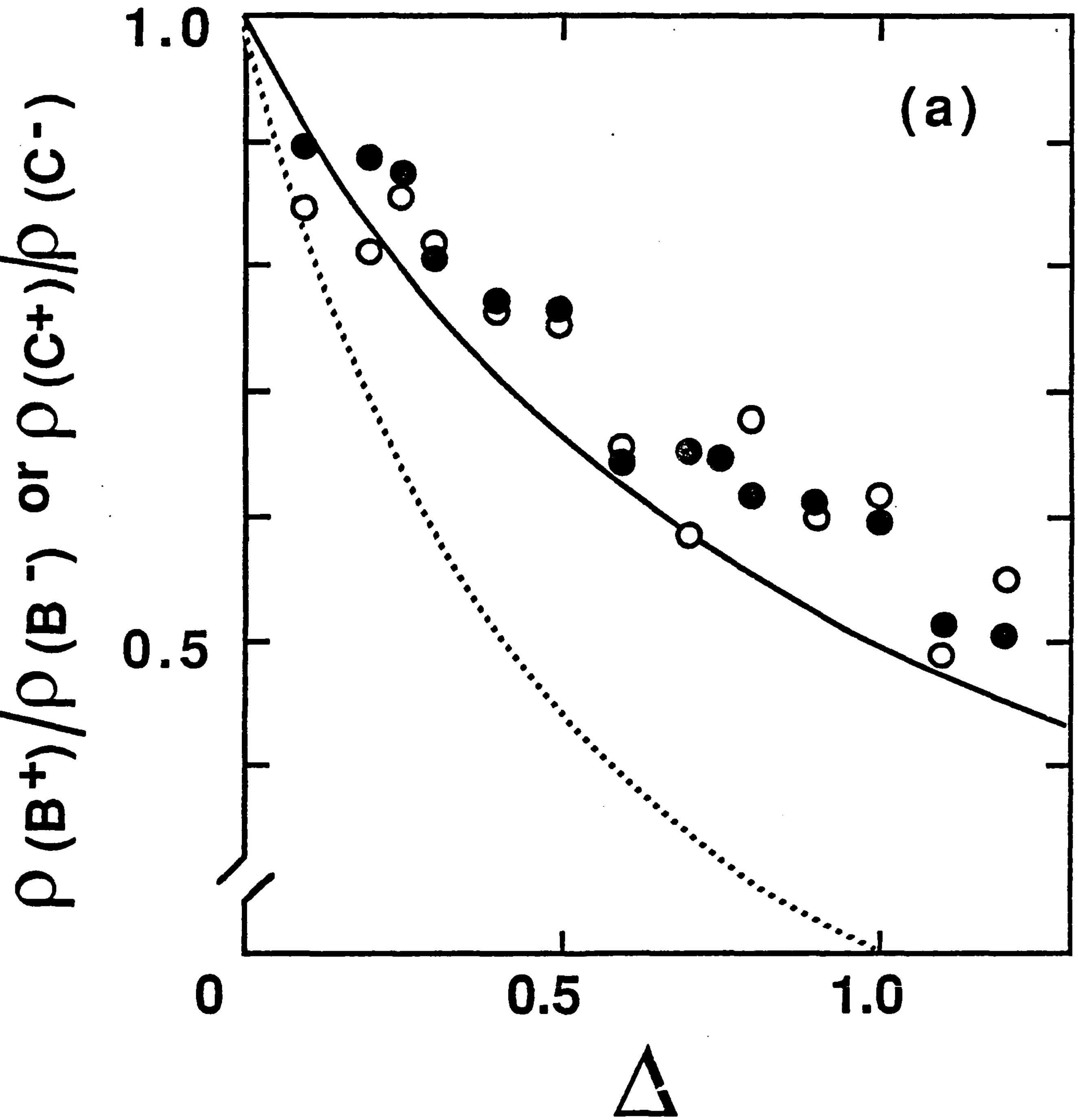
(a)
ADC
x
x^2
x
x^3
–
$x - x^3$
Data Analysis
$\rho_{(x)}$
$x(t)$
$\frac{1}{\tau_i}\int f(t)\,dt$
$f(t)$
+
comp x_1
x_1
comp x_0
x_0
AND
$g(x)$
x
Noise Gen & Filter $V_n(t,\tau)$


(b)
.50
.25
0
-.25
-.50
U(x)
A
B
C
D
-2.0
-1.5
-1.0
-0.5
0
0.5
1.0
1.5
2.0
X

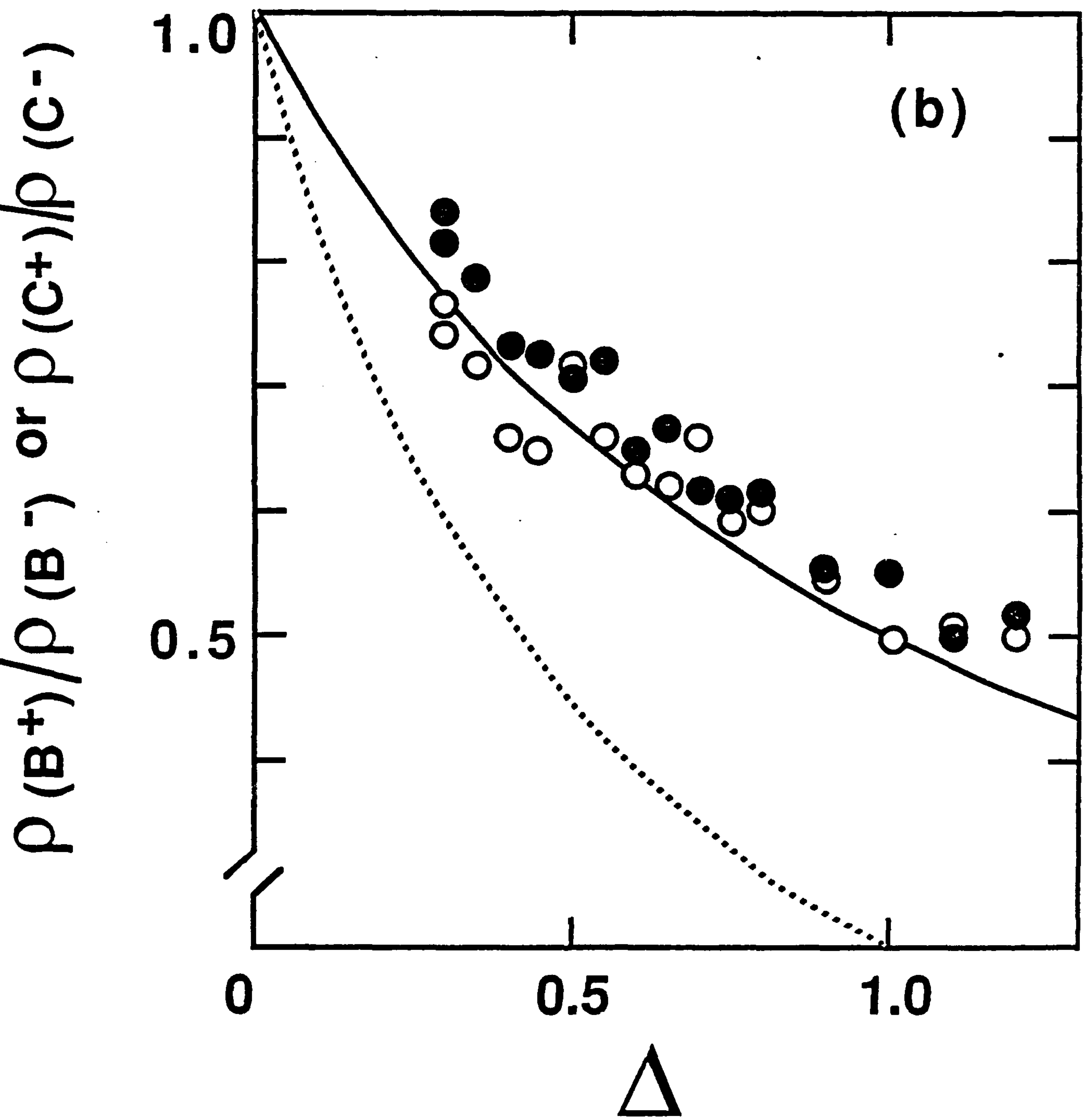




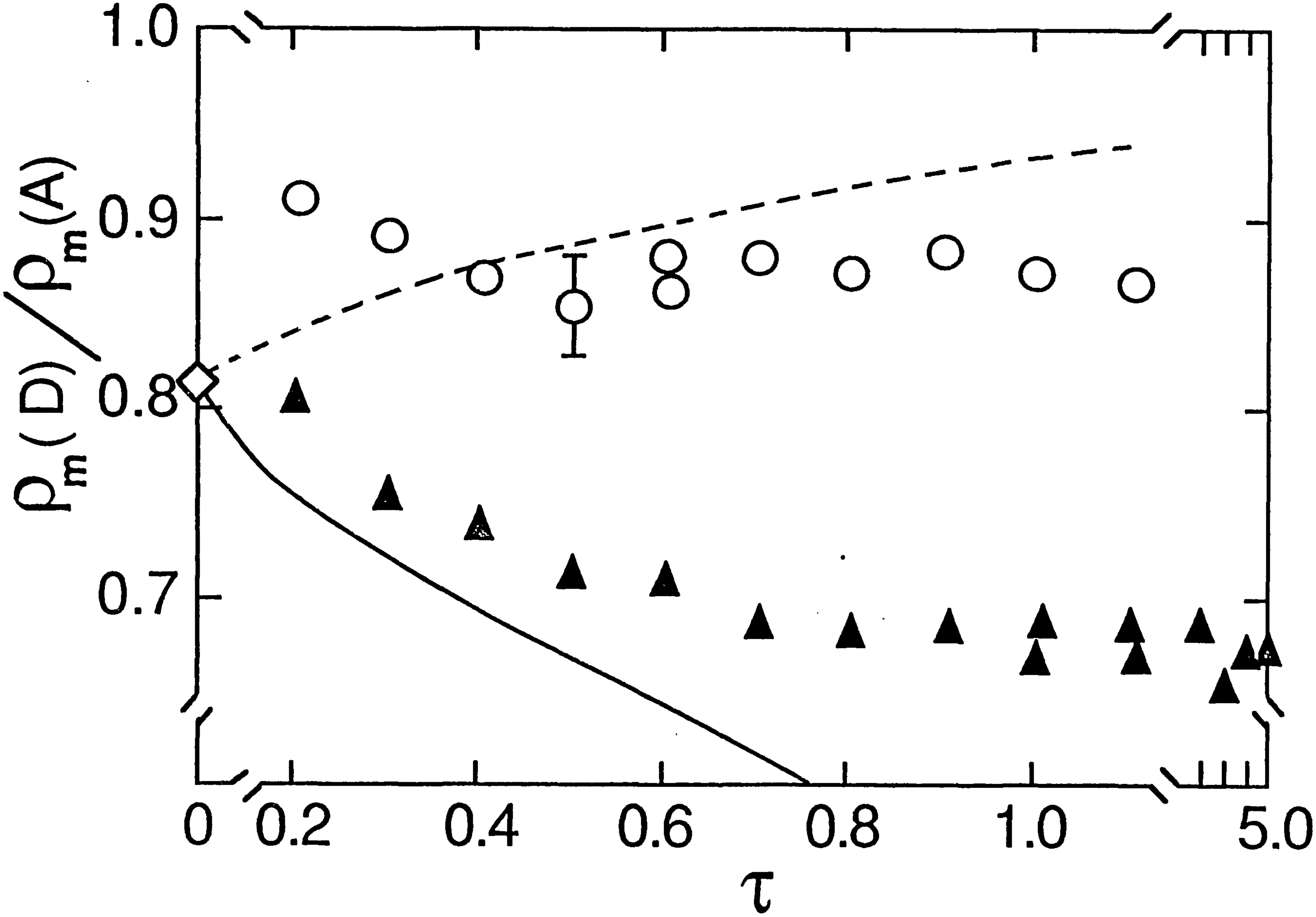
1.0
0.9
0.8
0.7
0
0.4
0.8
1.2
Δ
ρm(D)/ρm(A)



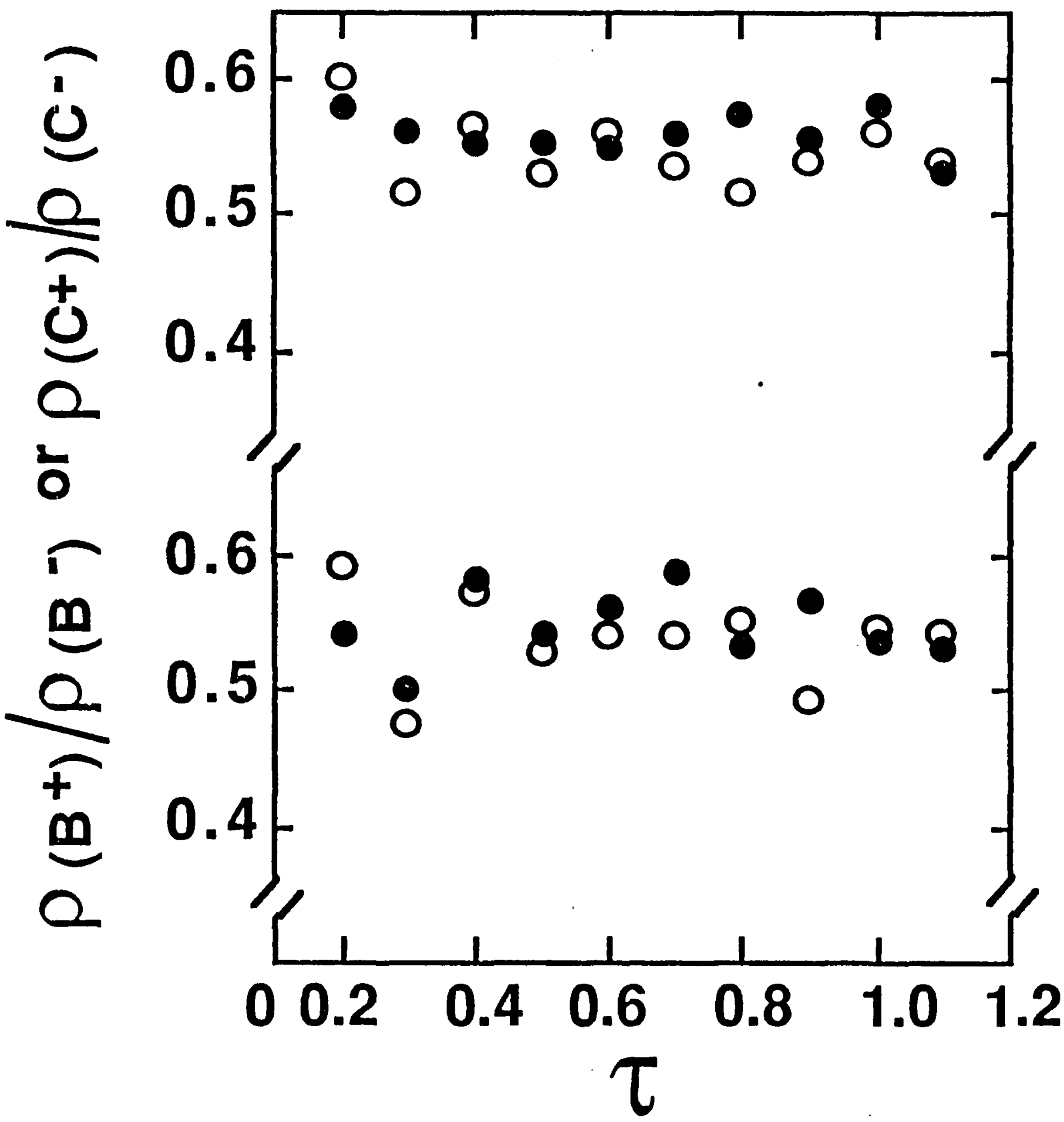
1.0
0.5
0
0.5
1.0
(a)
ρ (B+)/ρ (B−) or ρ (C+)/ρ (C−)
Δ



1.0
0.5
(b)
ρ (B+)/ρ (B-) or ρ (C+)/ρ (C-)
0
0.5
1.0
Δ



$\rho_m(D)/\rho_m(A)$
1.0
0.9
0.8
0.7
0
0.2
0.4
0.6
0.8
1.0
5.0
τ


ρ (B+)/ρ (B-) or ρ (C+)/ρ (C-)
0.6
0.5
0.4
0.6
0.5
0.4
0 0.2 0.4 0.6 0.8 1.0 1.2
τ

Quantum-noise quenching in correlated spontaneous-emission lasers as a multiplicative noise process: III. The role of noise color.

by

M. James* and Frank Moss

Department of Physics

University of Missouri at St. Louis

St. Louis, MO 63121

W. Schleich

Max-Planck Institut für Quantenoptik

D-8046 Garching bei München

Federal Republic of Germany, and

Center for Advanced Studies and Department of Physics and Astronomy

University of New Mexico

Albuquerque, NM 87131

H. Risken

Abteilung für Theoretsche Physik

Universität Ulm

D-7900 Ulm

Federal Republic of Germany

ABSTRACT

We show via an approximate, analytical technique and an analog simulation of the classical Langevin equation of a correlated, spontaneous emission laser (CEL) that noise of non-zero correlation time leads to an enhancement in the characteristic CEL noise quenching.

PACS 42.50.-p; 42.65.-k; 05.40.+j

The recognition that quantum noise determines the ultimate accuracy of active interferometers in applications involving, for example, gravitational wave detection[1-3], or ring-laser gyroscopes[4,5], has stimulated the investigation of laser systems with reduced spontaneous emission noise such as the correlated (spontaneous) emission laser[6-8] (CEL). In the CEL, the relative phase angle between two electromagnetic waves can be freed completely from the effects of spontaneous emission noise as a result of the fact that **this noise appears to the phase as a purely multiplicative stochastic process**[9]. The physics of noise quenching in the CEL has been discussed in parts I and II of this series of papers[10]. Moreover, it has been shown that, in a laser consisting of long-lived atoms, the **quantum, i.e. the spontaneous emission, noise is colored** instead of white.[11] An excellent discussion of both white and colored (pump) noise effects in lasers has been provided recently[12]. In this brief note, we demonstrate the purely classical quenching effect of colored, multiplicative noise in a system with a periodic potential as has been shown to apply to the CEL. Such effects have been previously studied theoretically for white, additive noise using continued fractions[13] and for white, multiplicative noise by approximate analytic techniques[18].

We do not intend to present here a theory of the CEL with long-lived atomic states, nor are we concerned with the intricacies of colored noise theory and the various approximation schemes associated therewith[14] as used to calculate approximate Fokker-Planck equations (except as in Ref. 22). Rather, we focus on the influence of noise color on the noise quenching effect as obtained from the classical Langevin equation of the two-mode CEL. A similar equation also describes the phase of the

electromagnetic field in a two-photon CEL[15].

We pursue this study following two strategies: First, we perform an approximate analysis of the Langevin equation by linearization within a physically relevant range of parameters. With the help of some standard techniques of noise theory[12], we find the (approximate) width of the steady state probability distribution, $P_0(\phi)$ for the relative phase ϕ, to be reduced by a factor $(1 + \gamma\tau)^{-1}$ compared to the white noise case. Here, τ is the noise correlation time and $\gamma = (b^2 - a^2)^{1/2}$, where b and a denote the laser gain and the detuning between the two waves respectively. Thus the CEL noise quenching is not only preserved in the case of colored noise but is even enhanced. Moreover, the white noise results[10] for the noise induced drift $<\Delta>$, and the noise induced asymmetry are both reduced by this factor. In the second approach, we simulate the Langevin equation of the CEL with an electronic circuit using a circuit and techniques similar to those previously described[16].. The simulator provides immediately the steady state distributions P_0 as shown by the examples in Fig. 1, for a range of values of τ. We compare and contrast these distributions to Gaussian approximations with first and second moments $<\Delta^{1,2}>$. We find qualitative agreement between the measured and calculated results and, in particular, we are able to confirm the predicted enhancement of the CEL noise quenching by color.

The natural extension of the CEL Langevin equation for the relative phase difference between the two modes in the presence of Gaussian noise,

$$<\epsilon(t)\epsilon(s)> = (D/\tau)\exp[-|t - s|/t], \tag{1}$$

of noise intensity D and correlation time τ with zero mean,

$$\langle\epsilon(t)\rangle = 0, \tag{2}$$

is[10,17,18]

$$\dot{\phi} = a + b\sin\phi + \epsilon(t)\sin\phi. \tag{3}$$

For small noise intensity $D/b \ll 1$, and for small detuning $|a|/b \ll 1$, we can try the *ansatz*[10,17]

$$\phi(t) \cong \pi + \arcsin(a/b) + \Delta(t), \tag{4}$$

where $|\Delta| \ll 1$. Thus Eq. (3) reduces to

$$\dot{\Delta} = -[\gamma + (\gamma/b)\epsilon(t)]\Delta - (a/b)\epsilon(t),$$

where γ was defined above in terms of a and b, and with the obvious solution

$$\Delta(t) = \Delta_0 \exp\left[-\gamma t - \frac{\gamma}{b}\int_0^t dt'\epsilon(t')\right]$$

$$-\frac{a}{b}\int_0^t dt'\epsilon(t')\exp\left[-\gamma(t - t') - \frac{\gamma}{b}\int_{t'}^t dt''\epsilon(t'')\right], \tag{5}$$

where $\Delta_0 = \Delta(t=0)$. From Eq. (5) it is straight forward to evaluate the moments[20] $\langle\Delta^j\rangle$ for $j = 1$ and 2.

With the help of Eqs (1) and (2) we can find the center-of-gravity, $<\phi>$, for the stationary distribution, at

$$<\phi> \cong \pi + \arcsin(a/b) + <\Delta>$$

$$\cong \pi + \arcsin(a/b) + (1 + \gamma\tau)^{-1}(a/b)(D/b), \tag{6}$$

and the approximate width of P_0 is governed by

$$\sigma^2 \equiv <\phi^2> - <\phi>^2 \cong (1 + \gamma\tau)^{-1}(a/b)^2(D/\gamma). \tag{7}$$

The Gaussian,

$$P_0^{app}(\phi) = (2\pi)^{-1/2}\sigma^{-1}\exp[-(2\sigma^2)^{-1}(\phi - \langle\phi\rangle)^2], \tag{8}$$

with $<\phi>$ and σ given by Eqs. (6) and (7), thus represents the simplest approximation to the exact distribution P_0, which we do not calculate here.

The noise source $\epsilon(t)$ is multiplicative[9] and so gives rise to a noise-induced asymmetry which manifests itself as a separation of the center-of-gravity of the right hand peak of P_0, given by Eq. (6), and the location of its maximum[22],

$$\phi_{max} \cong \pi + \arcsin(a/b) - (1 + \gamma\tau)^{-1}(a/b)(D/b). \tag{9}$$

This asymmetry is obviously not contained in the Gaussian approximation.

From Eqs. (6), (7) and (9), we recognize that the noise color modifies the white noise ($\tau = 0$) moments by the prefactor $(1 + \gamma\tau)^{-1}$, which signifies an enhancement of the noise quenching characteristic of the CEL. We emphasize that this noise color enhanced quenching is an entirely classical effect which may have applications to other systems. Nor does the quenching stem from the multiplicative nature of the noise in this particular application. For example, a similar result can easily be obtained for the standard phase locked ring laser Langevin equation[23],

$$\dot{\phi} = a + b\sin\phi + \epsilon(t). \tag{10}$$

For this case, following an analysis similar to the above results in an approximate width for the stationary distribution in the phase locked region given by,

$$\sigma^2_{\text{ring laser}} \cong (1 + \gamma\tau)^{-1}(D/b), \tag{11}$$

which shows that the width, for this simpler case of additive noise, is governed by the same color prefactor.

The techniques for simulating periodic potentials with analog circuits have been previously discussed[16], and here we use the same approach except for the following simple alterations: First, the hybrid analog-digital-analog system for producing the spatially periodic force was replaced with an analog chip[26] which accomplishes the same task; and second, an additional multiplier was used to implement the multiplicative noise. The voltages representing $\phi(t)$ were digitized into time series of approximately four million points from which the stationary probability densities were assembled. In contrast to the simulation of this same system with additive noise described in Ref. 16, in the present case only the one-dimensional densities were

measured. Our purpose is simply to demonstrate the phenomenon of noise quenching and the noise color effects in this purely classical multiplicative system.

Figure 1 shows a set of three probability densities measured for values of τ ranging from 0.1 (quasi-white noise) to 10 (strongly colored noise). These densities are consistentconsistant with the behavior predicted by Eqs. (6) and (9), which predict that the effect of the multiplicative noise is to induce a τ-dependent shift between the center-of-gravity and the maxima of the peaks of P_0. This shift is, however, quite small for small D and a where the theory should be accurate. For the values of the parameters shown in the Figure caption, the τ-dependent term in Eqs. (6) and (9), $(1 + \gamma\tau)^{-1}$, is only $\cong 0.12$ for $\tau = 0.1$ and $\cong 0.013$ for $\tau = 10$. These numbers must be compared with π and with the deterministic shift, arcsin(a/b), which is 0.52. On the scale of our measurements, the predicted shift is just observable, but too small to measure quantitatively. The quenching predicted by Eq. (7), $\sigma \cong 0.26$ for $\tau = 0.1$ and $\sigma \cong 0.09$ for $\tau = 10$, is also clearly evident and in rough agreement with the widths of the P_0 curves shown. Measurements for larger D ($\cong 1$) and larger a ($\rightarrow 1$, where free running commences in the deterministic case) show exaggerations of all these effects, however, the linearized theory cannot be expected to apply for these large parameter values, and so we do not show those data.

Also evident in Figure 1 is the hint of a third maximum in P_0 near $\phi = 0$. This is the location of a zero in the deterministic force. Furthermore this zero is a stable fixed point. In the case of pure white, multiplicative noise, the diffusion approaches zero, so that $P_0 \rightarrow \delta(\phi = 0)$ at this point. This maximum is easily observed using the simulator for small τ and larger D values. The relative amplitude of this maximum is observed to decrease with increasing values of τ, as the length scale of the noise correlated motions grows relative to the size of the deterministic basin of attraction

around $\phi = 0$.

In summary, we have estimated the magnitudes of the noise quenching and the color induced shifts between the center-of-gravity and the maximum of a peak in the density using an approximate, linear analysis which begins with the classical Langevin equation for the CEL. In addition, we have qualitatively observed in an electronic analog all the features forecast by the calculation.

We are grateful to Th. Leiber and K. Vogel for several stimulating discussions. We thank M. O. Scully for introducing us to the concept of the CEL and for his encouragement and support. This work was supported in part by the U. S. Office of Naval Research, grant number N00014-88-K-0084 and by NATO grant number 0770/85.

REFERENCES

*present address: Department of Physics; University of Wisconsin at Parkside; Kenosha, WI 53141

1. For the role of quantum noise in **passive** interferometers, see the seminal article: C. M. Caves, Phys. Rev. D **23**, 1693 (1981)

2. For a review of gravitational wave detection using Michelson interferometers, see for example: W. Winkler, G. Wagner and G. Leuchs in **Fundamentals of Quantum Optics II**, Lecture Notes in Physics, Vol. 282, edited by F. Ehlotzky (Springer Verlag, Berlin, 1987)

33. M. O. Scully and J. Gea-Banacloche, Phys. Rev. A **34**, 4043 (1986)

4. For a review on ring laser gyroscopes see, W. W. Chow, J. Gea-Banacloche, L. M. Pedrotti, V. E. Sanders, W. Schleich and M. O. Scully, Rev. Mod. Phys. **57**, 61 (1985)

5. (a) M. O. Scully, Phys. Rev. A **35**, 452 (1987)
   (b) J. Krause and M. O. Scully, Phys. Rev. A **36**, 1771 (1987)

6. (a) M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985)
   (b) M. O. Scully and M. S. Zubairy, Phys. Rev. A **35**, 752 (1987)
   (c) J. Bergou, M. Orszag and M. O. Scully, Phys. Rev. A **38**, 768 (1988)

7. S. Swain, J. Mod. Opt. 35, 1 (1988) and *ibid* 103 (1988)

8. For experimental realizations of CEL devices see:

(a) P. Toschek and J. Hall in **XI International Conference on Quantum Electronics Technical Digest Series, Vol. 21** (Optical Society of America, Washington, 1987) p. 102

(b) M. Ohtsu and K. -Y. Liou, Appl. Phys. Lett. 52, 10 (1988)

9. (a) A. Schenzle and H. Brand, Phys. Rev. A 20, 1628 (1979)

(b) W. Horsthemke and R. Lefever, **Noise Induced Transitions, Theory and Application in Physics, Chemistry and Biology** (Springer Verlag, Berlin, 1984)

10. (a) W. Schleich and M. O. Scully, paper I, Phys. Rev. A 37, 1261 (1988).

(b) W. Schleich, M. O. Scully and H.-G. von Garssen, paper II, Phys. Rev A 37, 3010 (1988).

11. M. O. Scully, G. Süssmann and C. Benkert, Phys. Rev. Lett. **60**, 1014 (1988).

12. (a) H. Risken, **The Fokker-Planck Equation, Method of Solutions and Applications, Vol. 18** Springer Series in Synergetics; edited by H. Haken (Springer Verlag, Berlin, 1984)

(b) P. Jung and H. Risken, Phys. Lett. **103A**, 38 (1984)

(c) P. Jung, Th. Leiber and H. Risken, Z. Phys. B **66**, 397 (1987)

(d) Th. Leiber, P. Jung and H. Risken, Z. Phys. B **68**, 123 (1987)

13. H. Risken, *loc. cit.* Ref. 12 (a), p. 292.

14. See for example the reviews by various groups in : **Noise in Nonlinear Dynamical Systems**, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, in press)

15. M. O. Scully, K. Wodkiewicz, M. S. Zubairy, J. Bergon, N. Lu and J. Meyer ter Vehn, Phys. Rev. Lett. 60, 1832 (1988); see also C. Benkert, W. Schleich and M. O. Scully, to be published

16. K. Vogel, H. Risken, W. Schleich, M. James, F. Moss and P. V. McClintock, Phys. Rev. A 35, 463 (1987); and references therein

17. For reasons related to the construction of the electronic circuit the phase ϕ used throughout this article differs from the phase of the earlier CEL work (Ref. 10) by a phase shift π, which does not introduce any change in the physics. Moreover, the factor 1/2 in the argument of the sine function in front of the noise in the two mode CEL, as shown in the previous CEL references, has been omitted for simplicity. From the linearization *ansatz*, Eq. (4), we recognize that it essentially redefines the noise intensity. Note, however, that the phase of the electric field of the two-photon CEL (Ref. 15) with memory effects obeys Eq. (3).

18. For a discussion of this equation in the case of white noise, see I. I. Fedechenia and N. A. Usova, Z. Phys. B 50, 263 (1983); *ibid* B 52, 69

(1983)

19. For an exact expression for the steady state distribution P_0 of the Fokker-Planck equation corresponding to Eq. (3),

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi}\{[a + (b + \epsilon)\sin\phi]P\} + \frac{1}{\tau}\frac{\partial(\epsilon P)}{\partial \epsilon} + \frac{D}{\tau^2}\frac{\partial^2 P}{\partial \epsilon^2},$$

in terms of (infinite) matrix continued fractions, see R. Habiger and H. Risken, to be published.

20. It is well known (Ref. 9) that linearization of a Langevin equation with multiplicative noise leads to divergent higher moments,

$$\left\langle \exp\left\{ n\left[-bt + \int_0^t dt'\epsilon(t') \right] \right\} \right\rangle = \exp[-n(b-nD)t],$$

where, for simplicity, we have assumed white noise. Obviously, this expression diverges for orders n > b/D. In this work, however, we confine the discussion to the first two moments only, which, over the range of parameters considered, do not diverge.

21. In Eq. (36) of paper I of this series (Ref. 10) the minus sign should be replaced by a plus sign.

22. The expression for ϕ_{max} can be found by first deriving the expression for

white noise. In the limit of vanishing probability current we find for ϕ_{max} (where P′ vanishes),

$$\phi_{max} = \pi + \arcsin(a/b) - (a/b)(D/b).$$

For small colored noise, using the celebrated Hanggi *ansatz*[24], we must replace D by

$$D(\tau) = D/[1 - \tau\langle(a + b\sin\phi)'\rangle]$$

$$\cong D/[1 - \tau b\cos\phi_0] = D/(1 + \gamma\tau),$$

where $\phi_0 = \pi + \arcsin(a/b)$, and where the prime denotes the spatial derivative.

23. (a) K. Vogel, Th. Leiber, H. Risken, P. Hanggi and W. Schleich, Phys. Rev. A **35**, 4882 (1987).

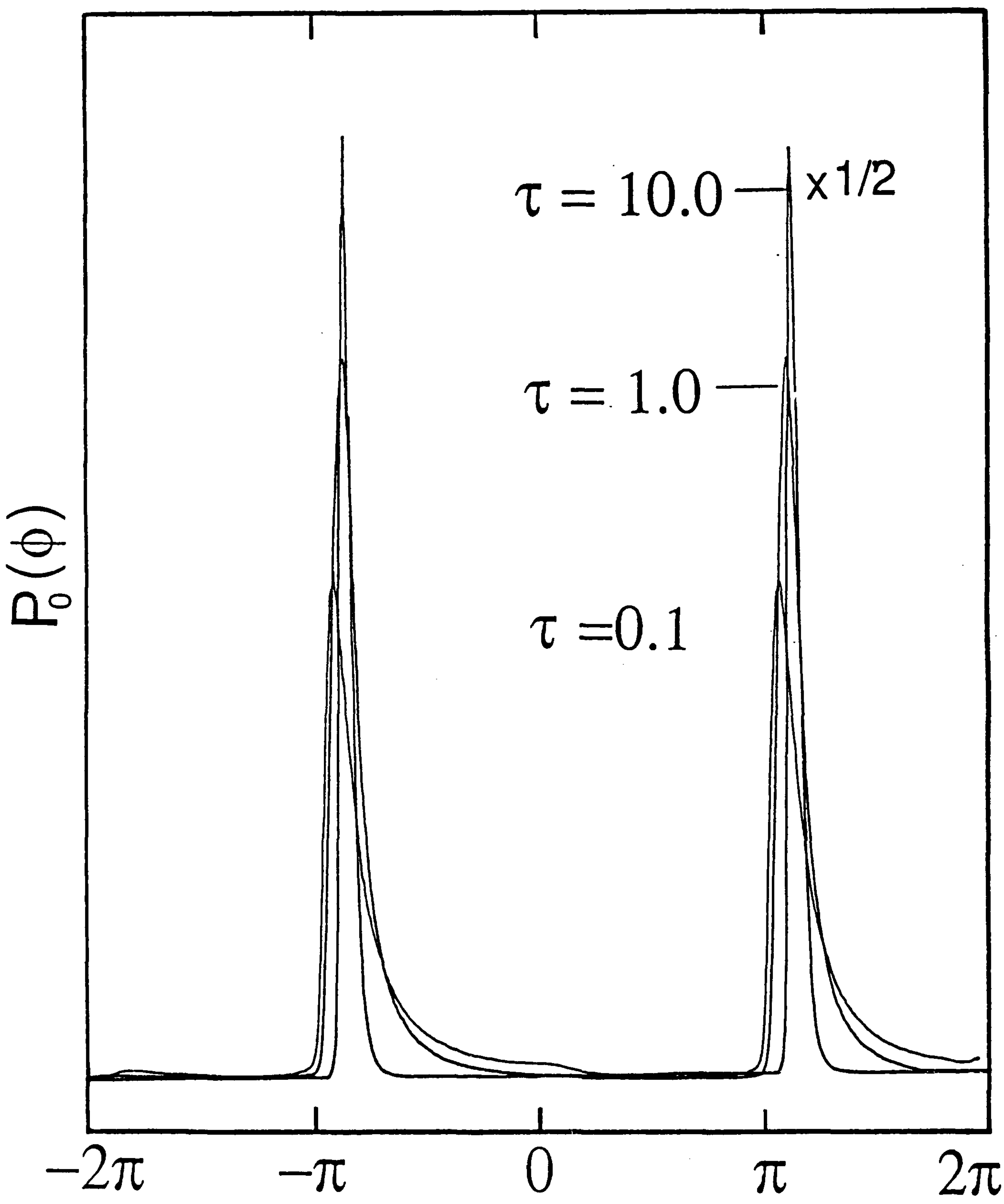
(b) K. Vogel, H. Risken and W. Schleich, in *loc. cit.* Ref.14.

24. P. Hanggi, T. J. Mroczkowski, F. Moss and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985)

25. Analog Devices model AD639.

FIGURE CAPTION

Fig. 1. Three probability densities measured for $D = 0.25$, $a = 0.5$ and $b = 1$. The values of τ are shown on the curves. The vertical scale is in arbitrary units, however the curve for $\tau = 10$ has been reduced in amplitude as shown.



$P_0(\phi)$
$\tau = 10.0$
x1/2
$\tau = 1.0$
$\tau = 0.1$
-2π
$-\pi$
0
π
2π